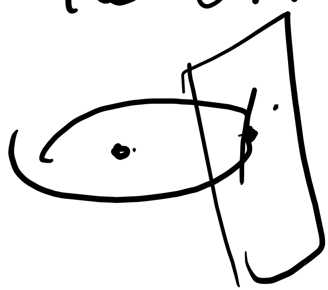
12/10/2021

Lecture-7

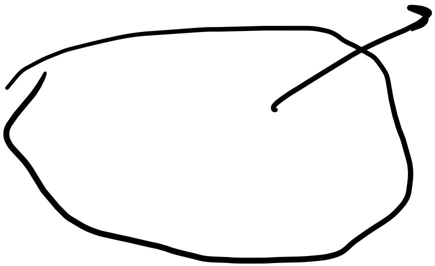
Theorem: Every line-free closed convex set $K \subseteq \mathbb{R}^n$ is the convex hull of its extreme points and extreme rays;

$$K = \mathrm{conv}\,(\mathrm{ext}(K) \cup \mathrm{extr}(K)).$$

Corollary: Every nonempty compact convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.



$$\mathrm{ext}(H \cap K) \subseteq \mathrm{ext}(K).$$



K

$\vec{x} \notin \underline{K}$

$\vec{x}$ $\quad K$

↳ closed

$\boxed{\underline{A}}\,\vec{x} \leq \boxed{\vec{b}} \longrightarrow$ polytope

↳ intersection of finitely many closed halfspaces.

<u>Proposition</u>: A polytope has finitely many extreme points.

<u>Proof</u>: $\underline{n = 1}$.

at max, there are two extreme points for a polytope in $\mathbb{R}$.

Assume ~~$\{ \vec{c_1} \cdot \vec{x} = d_1 \}$~~

$$P = \{ \vec{x} \in \mathbb{R}^n : \vec{a_i}^T \vec{x} \leq b_i, \ \vec{a_i} \neq 0, \ i = 1, \ldots, m, \ \vec{c_j}^T \vec{x} = d_j, \ j = 1, \ldots, p \}, \ \vec{c_j} \neq 0.$$

<u>Induction Hypothesis</u>:

If $\dim(K) \leq n-1$, then K has finitely many extreme points.

Since $\{\langle \vec{c_1}, \vec{x} \rangle = d_1\}$ has one dimension lower than $\mathbb{R}^n$.

$\mathcal{P} \subseteq \boxed{\{\langle \vec{c_1}, \vec{x} \rangle = d_1\}}$

By induction hypothesis, we are done.

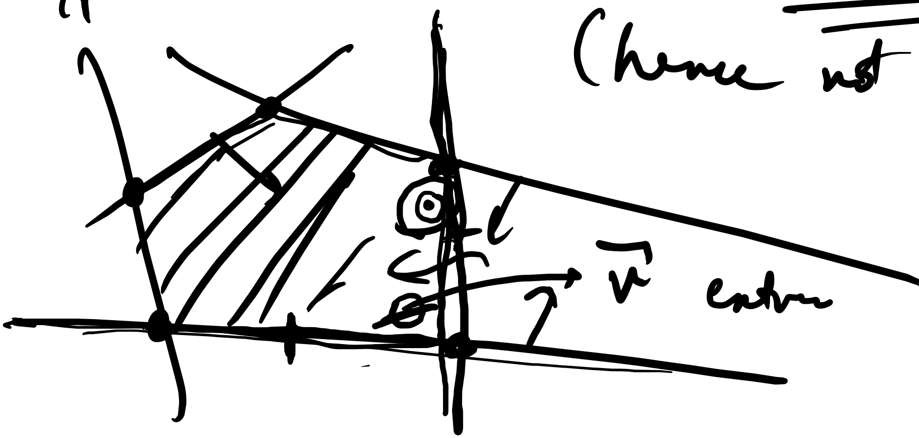
relint(K)

→ $K \subseteq \mathbb{R}^n$, either $\boxed{\mathrm{int}(K) \neq \phi}$ or $\boxed{\mathrm{int}(K) = \phi}$

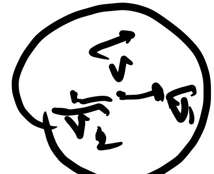
Claim. If $\vec{v}$ is an extreme point of $\mathcal{P}$, it must satisfy $\boxed{\langle \vec{a_j}, \vec{v} \rangle = b_j}$ for some j.

Proof of Claim

If $\vec{v}$ is in the interior of each of the finitely many supporting halfspaces, it must be in the interior of P. (hence not an extreme point.)



$$\vec{v}' = \frac{\vec{v}_1 + \vec{v}_2}{2}$$



$\vec{v} \neq \vec{v}_1$

$\vec{v} \neq \vec{v}_2$.

As $\vec{v}$ ~~satisfies~~ lies on the hyperplane

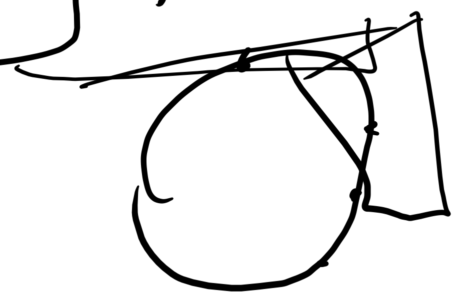
$H_j \leftarrow \boxed{\langle \vec{a_j}, \vec{x} \rangle = b_j}$, and $\vec{v}$ is

$\boxed{H_{\vec{a_j}, b_j}}$ an extreme point of $H_j \cap K$.

Thus there are finitely many choices for

$\vec{v}$.

$\boxed{H_{\vec{a_j}, b_j} \cap K}$ $j=1, \dots, m$

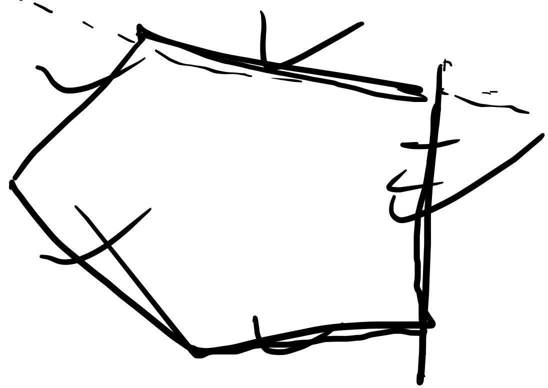


$\blacksquare$

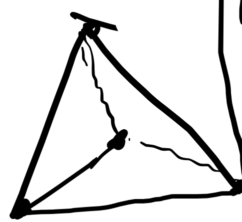
Remarks on the Combinatorial Theory of Polytopes:

(1) What is the largest number of $\boxed{\text{vertices}}$, edges, etc. of an n-dimensional polytope in $\mathbb{R}^n$ with a given number of facets?

$\mathbb{R}^2$



{ $(n-1)$-dimensional faces



Number of vertices = 4

Number of edges = 6

Number of facets = 4

(2) (Similar question) What is the smallest number of vertices, edges, etc ---- ---- ---- ?

For $n \geq 1$, the f-vector of an n-dim. polytope is the n-tuple.

$$f(P) = (f_0(P), f_1(P), \ldots, f_{n-1}(P)) \in \mathbb{Z}_+^n$$

$$P \to \vec{v} \in \mathbb{Z}_+^n$$

Qn. Is there a 3-dim polytope P such that

$f_0(P) = 7$ no. of vertices

$f_1(P) = 10$ no. of edges

$f_2(P) = 9$ no. of faces.

?

Ans. No. $7 - 10 + 9 = \boxed{6} \neq 2$.

$$\left[\sum_{j=0}^{n-1} (-1)^j f_j(P) = 1 + (-1)^{n-1} \right]$$

__Theorem__ (Upper Bound Theorem) McMullen.

$$f_0(P) \le \binom{k - \lfloor n/2 \rfloor - 1}{\lceil n/2 \rceil - 1} + \binom{k - \lceil n/2 \rceil}{\lfloor n/2 \rfloor}$$

$$= O\left(k^{\lceil n/2 \rceil}\right)$$

k $(n-1)$

Upper bound is achieved.

$m \quad F(\langle \vec{e}_i, \vec{x} \rangle) \leq k_j$

$$\boxed{\text{no of vectors} \leq \binom{m}{n}}.$$

Linear Programming:

General Form:

$$\text{minimize} \quad \vec{c}^T\vec{x} + \cancel{d}\, d \qquad \left(\vec{x} \in \mathbb{R}^n\right)$$

$\langle \vec{c}, \vec{x} \rangle$

$$\text{subject to} \quad G\vec{x} \preceq \vec{h}$$

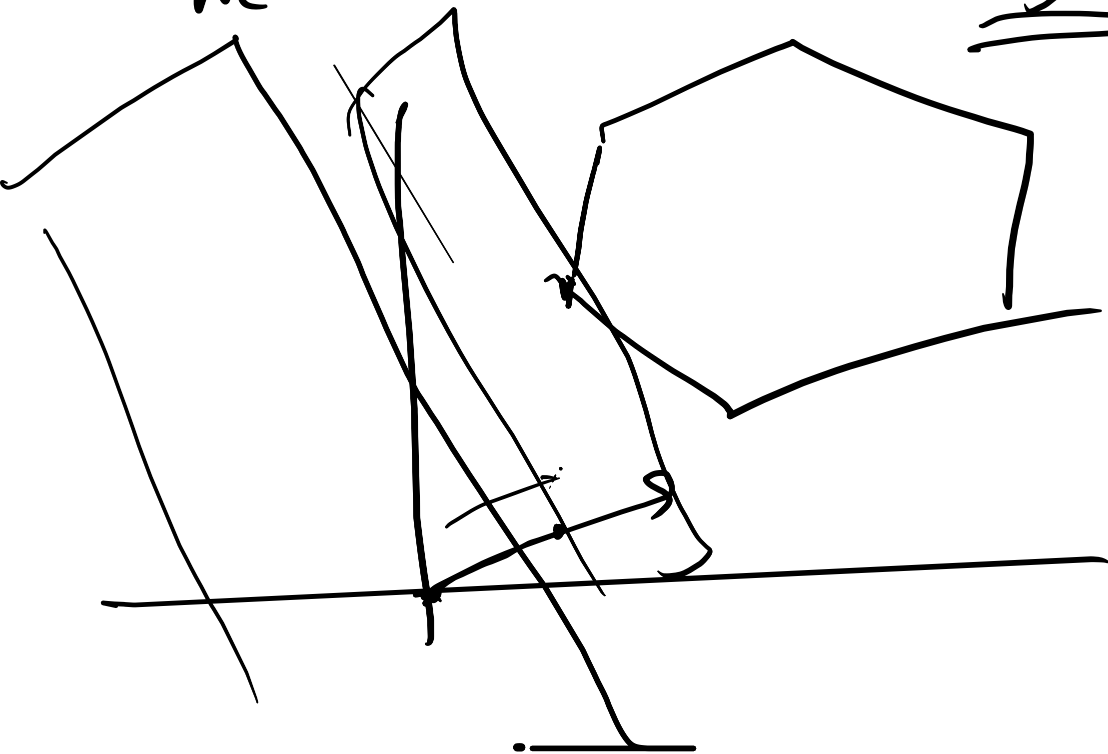
$$A\vec{x} = \vec{b}$$

where $G \in M_{m,n}(\mathbb{R})$, $A \in M_{p,n}(\mathbb{R})$, $\vec{h} \in \mathbb{R}^m$, $\vec{b} \in \mathbb{R}^p$.

* We may omit d in the objective function since it does not affect the decision space / feasible set.

* Feasible set of LP is a polyhedron $\mathcal{P}$.

We want to minimize $\langle \vec{c}, \vec{x} \rangle$ on $\mathcal{P}$.



$\langle \vec{c}, \vec{x} \rangle \leq \alpha$

Standard form of LP:

minimize $\langle \vec{c}, \vec{x} \rangle$

subject to $\boxed{A\vec{x} = \vec{b}}$

$\vec{x} \geq \vec{0}$.

(every decision variable is sign-constrained and only equality constraints are allowed)

(How to solve $A\vec{x} = \vec{b}$ — Gaussian elimination)

Canonical form of LP:

minimize $\vec{c}^T \vec{x}$ $(\langle \vec{c}, \vec{x} \rangle)$

subject to $A\vec{x} \geq \vec{b}$

$\vec{x} \geq \vec{0}.$

(every decision variable is sign-constrained and only inequality constraints are allowed.)

* The general form, standard form, canonical form of LP are equivalent to each other.
* Clearly an LP in standard form or canonical form is also in general form.

1. General $\longrightarrow$ standard.

Must eliminate inequality constraints and variables that are not sign-constrained.

a) Let $\boxed{\langle \vec{d}_1, \vec{x} \rangle \geq r_1}$ be an inequality constraint.

Create a new "surplus variable" and ~~rewr~~ $[0, s_1]^T$ $s_1 = \langle \vec{d}_1, \vec{x} \rangle - r_1$,

$s_1 \geq 0$ on the feasible set

$s_1 - \langle \vec{d}_1, \vec{x} \rangle = r_1 \Longleftrightarrow [-d_1 \cdots -d \;\; 1] \begin{bmatrix} x_1 \\ \vdots \\ s_1 \end{bmatrix}$

b. If x_j is not sign-constrained, create two new variables x_j^+, x_j^- and replace x_j with $x_j^+ - x_j^-$, and add the sign constraints $x_j^+ \geq 0$, $x_j^- \geq 0$.

$$\begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{m1} & \cdots & d_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$$

$$\begin{bmatrix} d_{11} & \cdots & d_{1n} & -1 & 0 & \cdots & 0 \\ \vdots & & & 0 & -1 & \cdots & 0 \\ d_{m1} & \cdots & d_{mn} & 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$$

we are adding on surplus variables

Exercise: Show that LP in standard form is equivalent to the LP in general form, in the sense that, ~~there~~ a feasible optimal solution to one gives a feasible optimal solution of the other.

$x_1 + x_2 = 1$

$x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 1$

2 - General $\longrightarrow$ canonical

Must eliminate equality constraints, and variables that are not sign-constrained.

a - Let $\langle \vec{d_i}, \vec{x} \rangle = r_i$ be an equality constraint: Replace it with a pair of inequality constraints: $\langle \vec{d_i}, \vec{x} \rangle \geq r_i$, $\langle -\vec{d_i}, \vec{x} \rangle \geq -r_i$.

b- If x_j is not sign-constrained, create two new variables x_j^+ and x_j^- and replace x_j everywhere by $x_j^+ - x_j^-$, and add the sign constraints $x_j^+ \geq 0$, $x_j^- \geq 0$.

* All forms ~~are~~ of LP are "equal", but some are ~~ap~~ more equal than others.

(Our LP from now on, will be written as LPS).

Exclusively use the standard form of LP for the simplex Method.

BASIC SOLUTIONS:

Def. Consider the LP

$$\min \langle \vec{c}, \vec{x} \rangle \text{ subject to } A\vec{x} = \vec{b};\ \boxed{\vec{x} \geq \vec{0}}$$

(LPS) where A is an $m \times n$ matrix of rank m.

$$A\vec{x} = \vec{b}$$

$$\vec{x} = \vec{b}'$$

$$\mathcal{L}(A\vec{x} = \vec{b}) = \phi$$

these columns correspond to free vars.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$x_3 \to t$

$x_1 + 2t = 1$

$\Rightarrow \quad x_1 = 1 - 2t$

$x_2 = 2 - 3t$

Think of $\vec{Ax}$ as

$x_1 A_1 + \dots + x_n A_n$

where, $A_j \to j^{th}$ column of A

$\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ = column span of A.

If columns $j_1 < j_2 < \dots < j_m$ are a basis

$$\begin{array}{cccc} 1 & 2 & \cdots & n \end{array}$$
$$\left[\begin{array}{cccc} \; & \; & \; & \; \end{array}\right]$$

that is, $A_{j_1}, \dots, A_{j_m}$ are linearly independent,

the corresponding **basic solution** (w.r.t. $j_1, \dots, j_m$) $\vec{x}$ is defined as follows.

$$\begin{array}{ccc} j_1 & \cdots & j_m \end{array}$$
$$\left[\begin{array}{ccc} \; & \; & \; \end{array}\right]$$

$B = [A_{j_1} | A_{j_2} | \dots | A_{j_m}]$

$\vec{x}_J = B^{-1}\vec{b}$ and $x_i = 0$ if $i \notin J$

$J = \{j_1, \dots, j_m\}$.

$$A\vec{x} = A_J\vec{x}_J + A_{J^c}\vec{x}_{J^c} = \vec{b}$$

$$\vec{x}_{J^c} = \vec{0}$$

$$\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{x_1A_1 + x_2A_2} + \boxed{x_3A_3}$$

$$J = \{1,2\} \qquad x_1A_1 + x_2A_2 = \begin{bmatrix} A_1 \mid A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_j = 0, \text{ for } j \in J^c.$$

$$+ x_3 A_3 \, [x_3]$$

$$A_J\vec{x}_J = \vec{b}$$

$$\Rightarrow \vec{x}_J^* = A_J^{-1}\vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_1 \mid A_2 \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

Any variable x_ℓ where $\ell \notin \{j_1, \dots, j_h\} = J$ is set to 0 and is called <u>nonbasic</u> w.r.t. J.

$x_{j_k} = k^{th}$ component of $\boxed{A_J^{-1} \vec{b}}$

and is known as <u>basic</u> ; the basic variables may or may not be zero.

(* The basic solution $\vec{x}$ is the unique vector (w.r.t. J) satisfying $\boxed{A\vec{x} = b}$ subject to the constraint $\{x_\ell = 0\}$ for $\ell \notin J$ })

If $\vec{x}_J$ satisfies the sign constraints $\boxed{\vec{x}_J \geq \vec{0}}$,

$\vec{x}$ is a **basic feasible solution** or **bfs**

(w.r.t. J)

* If $A \in M_{m,n}(\mathbb{R})$ has rank m, then

LPS always has a basic solution

$A\vec{x} = \vec{b}$

__Goal__: If the feasible region is nonempty, ~~Eh~~ when does LPS ~~most~~ have a __basic feasible solution__ (Strong form)

__Theorem__. In LPS, suppose that $\vec{w} \in \mathbb{R}^n$.
Then $\vec{w}$ is a vertex (or extreme point) of the feasible set F if and only if $\vec{w}$ is a basic feasible solution of the LPS instance.

Proof. ($\Rightarrow$)

If $\boxed{\vec{w} \text{ is a vertex of } F}$, $\vec{w}$ is feasible.

Let $\boxed{I = \{ i : w_i > 0 \}}$

$\boxed{\textbf{Claim.} \ \{ A_j : j \in I \} \text{ is linearly independent in } \mathbb{R}^m.}$

Proof of Claim. Suppose $\{ A_j : j \in I \}$ are linearly dependent. Then $\exists\, \vec{z} \in \mathbb{R}^n \setminus \{ \vec{0} \}$ such that

$$\sum_{j \in I} z_j A_j = 0, \quad \text{and} \quad \boxed{z_j = 0, \text{ if } j \notin I.}$$

Thus $\boxed{A\vec{z}} = \vec{0}$ ✓

$z_1 A_1 + z_2 A_2 \cdots + z_n A_n$

Recall. For $j \in I$, $w_j > 0$.

$$\theta = \min_{z_j \neq 0} \frac{w_j}{|z_j|}. \quad \text{Clearly } \theta > 0.$$

✓ $\vec{w}^+ = \vec{w} + \theta\vec{z}$, $\vec{w}^- = \vec{w} - (\theta\vec{z})$

$A\vec{w}^+ = A\vec{w} + \theta A\vec{z} = A\vec{w} = \vec{b}$

$A\vec{w}^- = \vec{b}$

Want to show: $\vec{w}^+ \geq \vec{0}$, $\vec{w}^- \geq 0$

For $j \notin I$, $\vec{w}_j^{\pm} = w_j \pm 0$

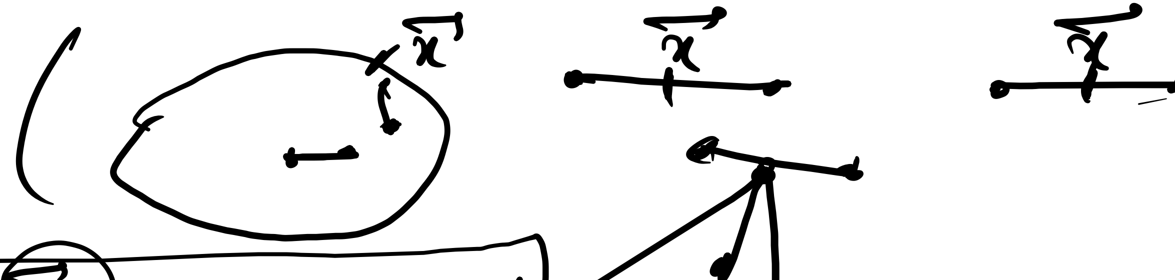
Thus, $\theta |z_j| \leq \frac{w_j}{|z_j|} \cdot |z_j| = w_j$.

$\Rightarrow w_j \pm \theta z_j \geq 0, \ \forall j \in I$

$\theta \vec{z} \neq \vec{0}$.

$\vec{w} = \frac{1}{2}(\vec{w}^+ + \vec{w}^-)$, and $\vec{w} \neq \vec{w}^+$, $\vec{w} \neq \vec{w}^-$

Contradiction as $\vec{w}$ is assumed to be an extreme point!!

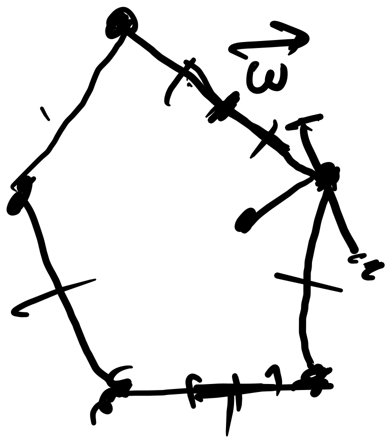


If $\frac{\vec{x}_1+\vec{x}_2}{2} = \vec{x}$, $\vec{x}_1, \vec{x}_2 \in K$

midpoint of line segment

$\{\vec{x}_1, \vec{x}_2\}$ → degenerate line segment

then $\vec{x} = \vec{x}_1 = \vec{x}_2$ → degen



5 extreme pow