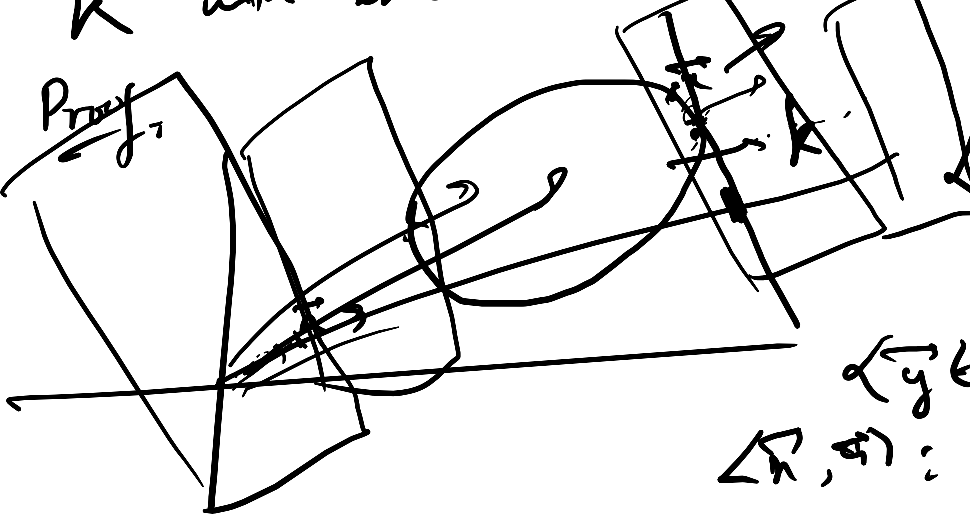
07/10/2021 # Lecture - 6

Theorem: If K is a nonempty compact convex set in $\mathbb{R}^n$, then to each vector $\vec{a} \in \mathbb{R}^n \setminus \{\vec{0}\}$ there is a support plane to K with exterior normal vector.

Proof:



$\{\vec{x} \in \mathbb{R}^n : \langle \vec{a}, \vec{x} \rangle = b\} \to H_{\vec{a}, b}$

$\{\vec{y} \in \mathbb{R}^n : \langle \vec{y}, \vec{a} \rangle = \langle \vec{x}, \vec{a} \rangle\}$

$\langle \vec{x}, \vec{a} \rangle = \sup \{\langle \vec{z}, \vec{a} \rangle : \vec{z} \in K\}$

$f_0(\vec{x}) = \vec{c}^{\,T} \vec{x} = \langle \vec{c}, \vec{x} \rangle$

Lemma: If $K \subseteq \mathbb{R}^n$ is a nonempty closed convex set with $\boxed{K \not\subseteq \operatorname{conv}(\operatorname{relbd} K)}$ then K is either an affine set or halfspace of an affine set.

Proof. Assume $\dim K = n$.

Let $\vec{x} \in \underline{\operatorname{int}(K)}$ with $\vec{x} \notin \operatorname{conv}(\operatorname{bd} K)$.

$K = \underline{\operatorname{int} K} \cup \boxed{\operatorname{bd} K}$

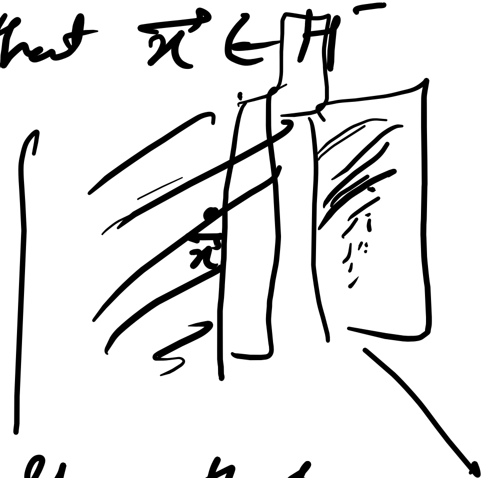
$\exists\, \vec{x} \in K$ s.t. $\vec{x} \notin \operatorname{conv}(\operatorname{bd} K)$

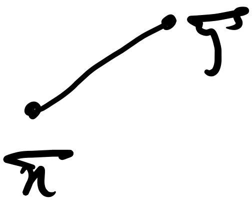
By the separation theorem, there is a closed halfspace such that $\vec{x} \in H^-$

and $\boxed{\text{conv}(\text{bd } K)} \subset H^+$

Let $\vec{y} \in \text{int}(H^+)$

$\cdot\, \vec{y}$



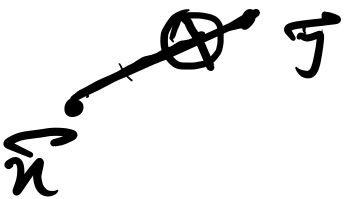


Show that conv(bd K) is a closed set ?

is closed convex

$[\vec{x}, \vec{y}]$

If possible, assume $\vec{y} \notin \text{int } K$



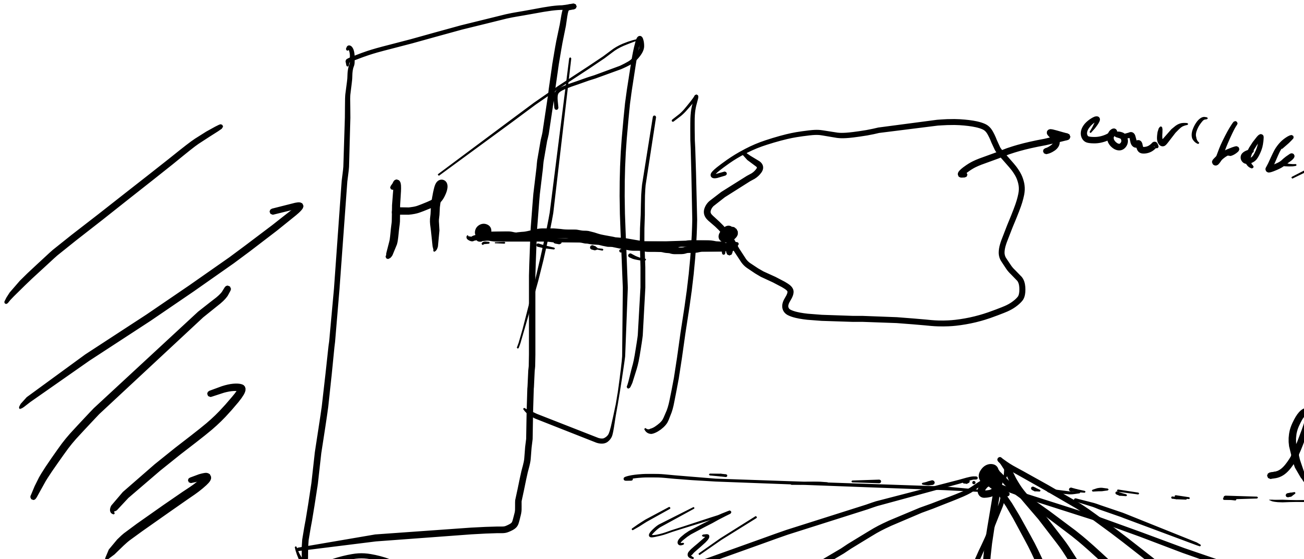
$K \cap \underline{\underline{[\vec{x}, \vec{y}]}}$ is closed.

$\subseteq \mathrm{int}(H^-)$.

Contradiction!!..

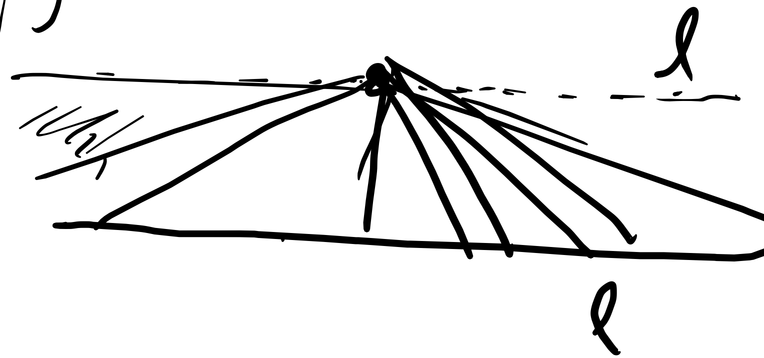
Say $\vec{y} \in \mathrm{int}(K)$.

$\Rightarrow \quad \mathrm{int}(H^-) \subseteq \mathrm{int}(K)$.



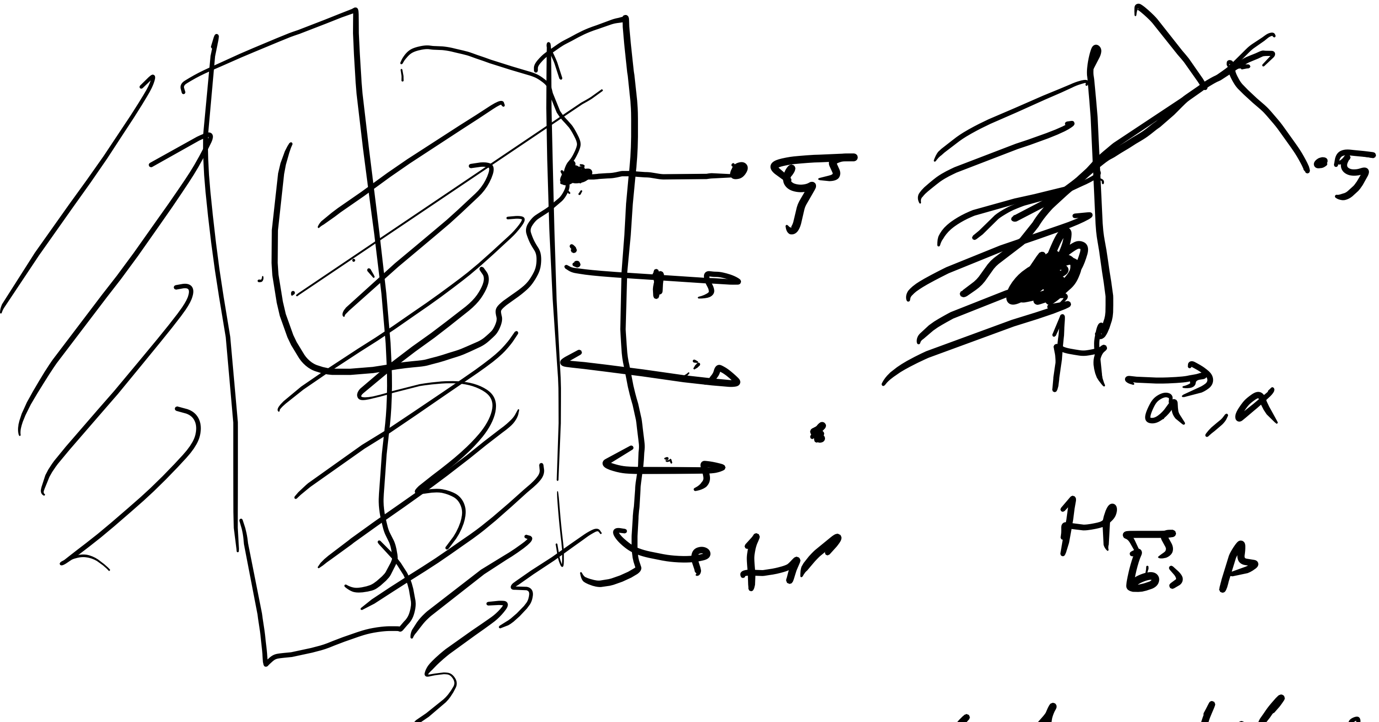
$H^- \subseteq K$

$$\boxed{K = \mathbb{R}^n}$$



If $K \neq \mathbb{R}^n$, $\exists\ \vec{y} \in \mathbb{R}^n$

s.t. $\vec{y} \notin K$.

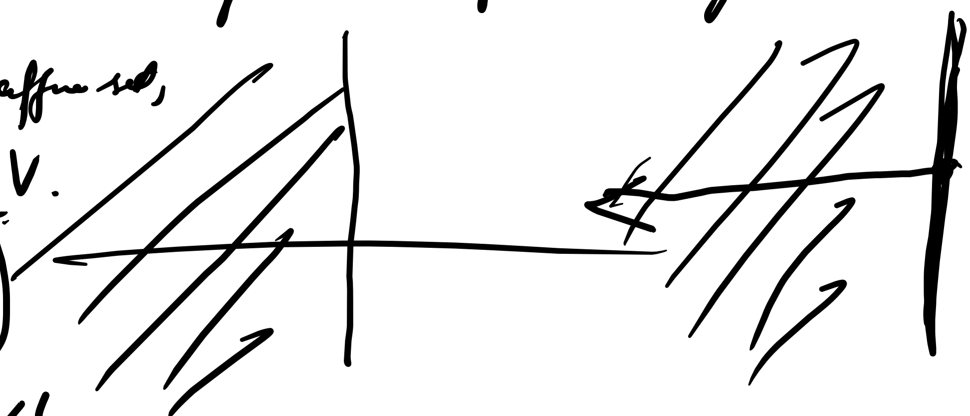


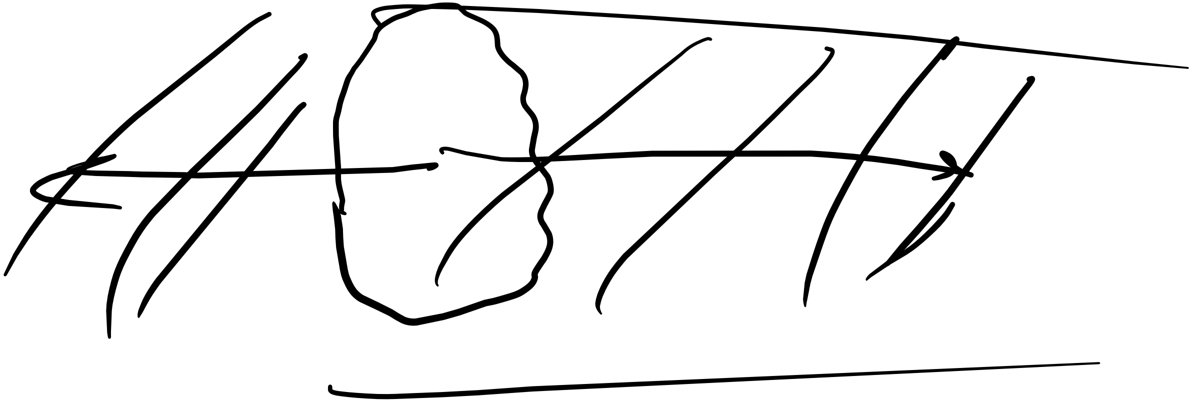
If H' is not parallel to H, H' intersects H. Hence, it cannot be a supporting plane of K.

Lemma – Each closed convex set $K \subseteq \mathbb{R}^n$ can be represented in the form $\boxed{K'} \oplus V$ where V is a linear space of $\mathbb{R}^n$ and K' is a line-free closed convex set in a subspace complementary to V.

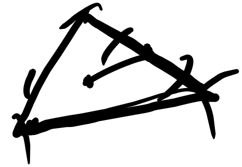
If $\boxed{K}$ is an affine set, what are K' & V.

$K = V + \underbrace{\boxed{\{x_0\}}}_{K'}$





Defn: Let $K \subseteq \mathbb{R}^n$ be a convex set.
A face of K is a convex set $F \subseteq K$ such that each segment $[\vec{x}, \vec{y}] \subseteq K$ with $F \cap \operatorname{relint}([\vec{x}, \vec{y}]) \neq \emptyset$, then $[\vec{x}, \vec{y}]$ is contained in F.

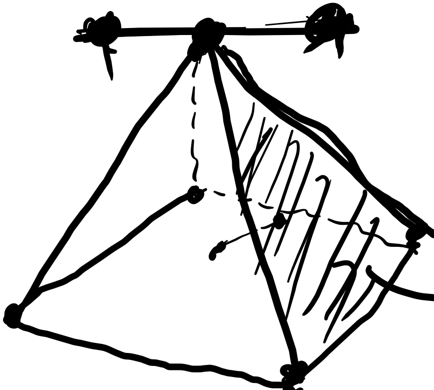


Equivalently, $\vec{x}, \vec{y} \in K$ and $\dfrac{\vec{x}+\vec{y}}{2} \in F$

implies $\vec{x}, \vec{y} \in F$.

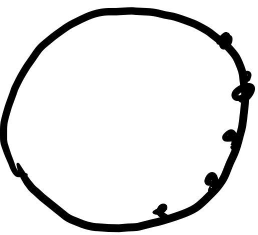
$\in \operatorname{relint}([\vec{x}, \vec{y}])$

$\dfrac{p}{2^k}$



triangle is a f

vertex 0-dimensional face.



If $\boxed{\langle \vec{z} \rangle}$ is a face of K, then $\vec{z}$ is called an <u>extreme point</u> of K.

$\Updownarrow$

In other words, $\vec{z}$ is an extreme point of K if and only if it cannot be written in the form $\vec{z} = \lambda \vec{x} + (1-\lambda)\vec{y}$ with $\vec{x}, \vec{y} \in K$, $\lambda \in (0,1)$ and $\vec{z} \neq \vec{x}$.

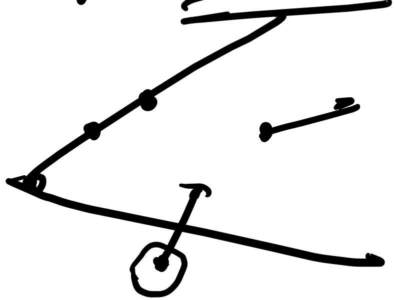
$\Updownarrow$

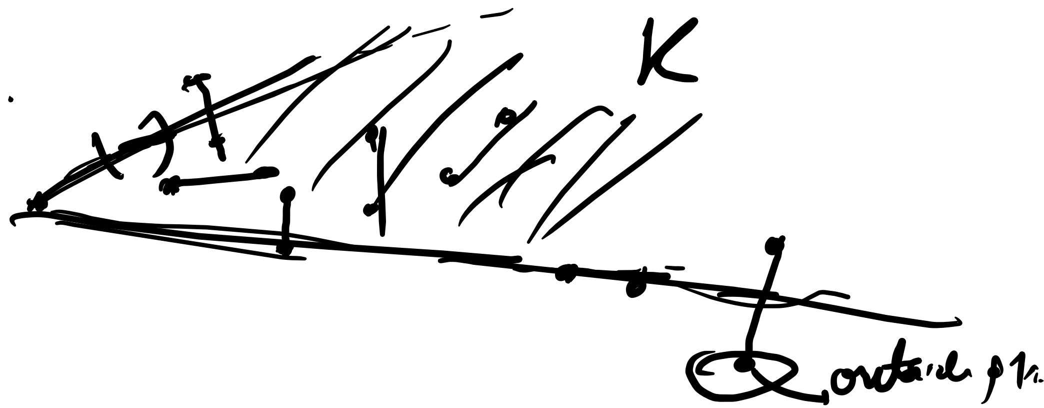
If $\vec{z} = \boxed{\lambda \vec{x} + (1-\lambda)\vec{y}}$ for some $\lambda \in (0,1)$ (for arbitrary $\vec{x}, \vec{y} \in K$)

then $\boxed{\vec{z} = \vec{x} = \vec{y}.}$

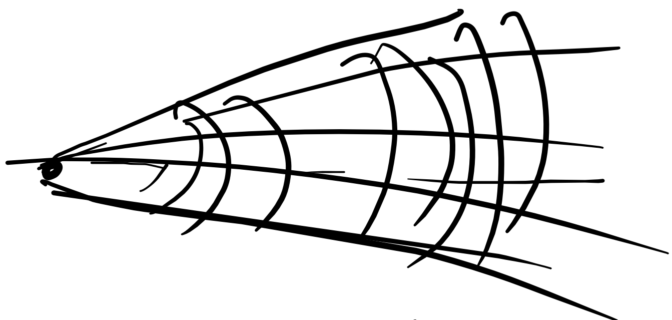
The set of all extreme points of K is denoted by <u>$\mathrm{ext}(K)$</u>.

An <u>extreme ray</u> of K is a ray that is a face of K. The set of all extreme rays ~~are~~ is denoted by <u>$\mathrm{extr}(K)$</u>.





Any line segment in K ~~is~~ is either contained in F or trivially intersect K



$\mathrm{extr}(K)$

is an infinite set.

in $\mathbb{R}^3$

& $\#(\mathrm{extr}(K)) \geq 4$

Main Theorem: Each line-free closed convex set $K \subseteq \mathbb{R}^n$ is the convex hull of its extreme points and extreme rays.

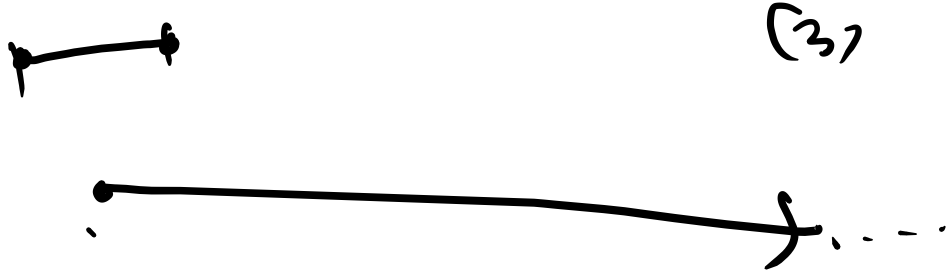
$$K = \operatorname{conv}(\operatorname{ext}(K) \cup \operatorname{extr}(K)).$$

Proof. For $n = 1$, what are the line-free closed convex sets?

(1) point

(2) line segment

(3) rays



Assume $n \geq 2$ and $\dim(K) = n$ and the assertion has been proved for closed convex sets in lower dimensions.

Since K is neither $\mathbb{R}^n$ nor a closed halfspace,

$$\boxed{K = \operatorname{conv}(\operatorname{bd} K)}.$$

By the support theorem,
each point $\bar{x} \in bd(K)$ lies on some
support plane, H, of K.

$H \cap K$ is a closed convex set,
of dimension strictly lower than n.

By the induction hypothesis $\vec{x}$ is in the
convex hull of the <u>extreme points</u> of $H \cap K$
and the <u>extreme rays</u> of $H \cap K$.

* Any extreme point of $H \cap K$ is in fact an extreme point of K.
* Any extreme ray of $H \cap K$ is in fact an extreme ray of K.

($\vec{x} \in H \cap K$).

If a line segment in K contains $\vec{x}$ in its relative interior, then it must be in $H \cap K$.

Corollary : (Minkowski's Theorem)

[Every compact convex set K in $\mathbb{R}^n$ is the convex hull of its extreme points]

Summary : $V \oplus$ ~~ext~~ conv (ext(K)) = $\text{extr}(K)$.

If $\boxed{\text{conv}(S)} = K$, then S contains all extreme points of K.

Proposition: A polytope has finitely many extreme points.