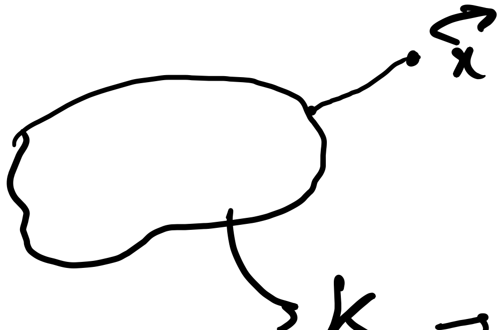
05/10/2021 Lecture - 5



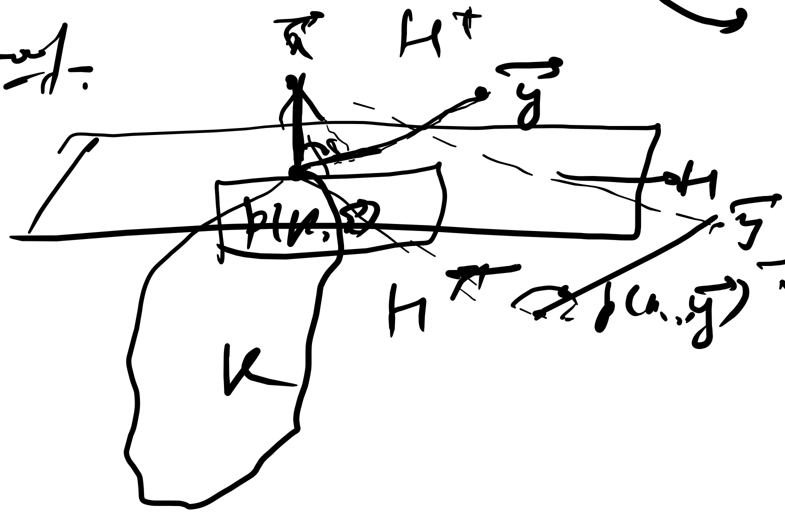
$K \to$ closed convex set on $\mathbb{R}^n$

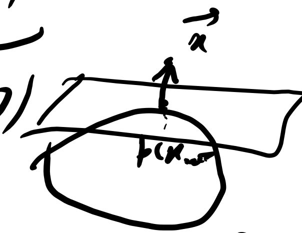
$[\, P(K, \cdot) : \mathbb{R}^n \longrightarrow K \,]$ (metric projection)

Lemma: Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $\vec{x} \in \mathbb{R}^n \setminus K$. The hyperplane $\boxed{H_{x}}$ through $p(K,\vec{x})$, orthogonal to $u(K,\vec{x})$, support K.

$$u(K,\vec{x}) = \frac{\vec{x} - p(K,\vec{x})}{d(K,\vec{x})}$$

Proof:





$$\boxed{\langle \vec{v}, \vec{w} \rangle > 0.}$$

$$\|\vec{v} - u\varepsilon\vec{w}\|_2 < \|\vec{v}\|$$

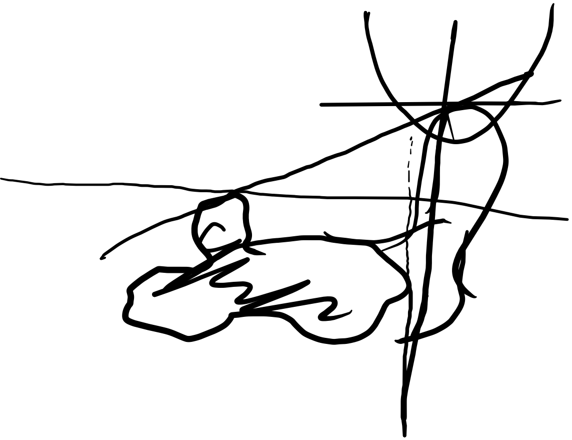
$\exists \varepsilon > 0$

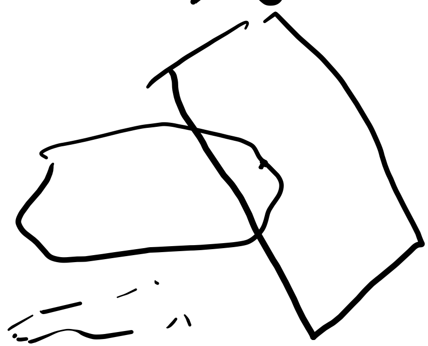
Equivalently, $\langle \vec{v} - \varepsilon \vec{w}, \vec{v} - \varepsilon \vec{w} \rangle$

$< \langle \vec{v}, \vec{v} \rangle$

$\Rightarrow \quad -2\varepsilon \boxed{\langle \vec{v}, \vec{w} \rangle} + \varepsilon^2 \boxed{\langle \vec{w}, \vec{w} \rangle} < 0$

> 0



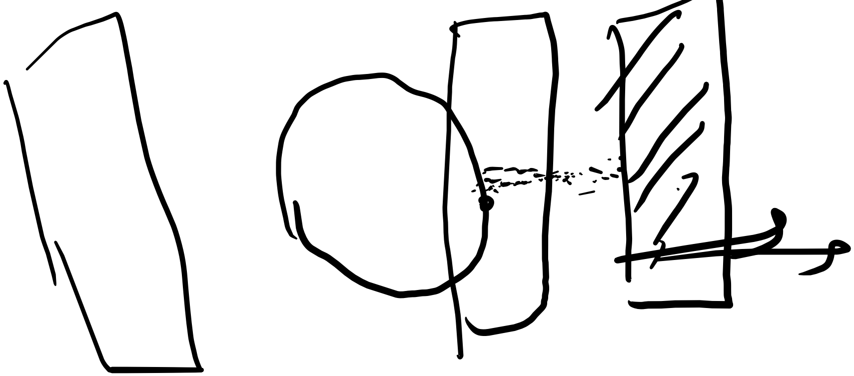


$H_{\vec{a}, b} = \{ \vec{x} \in \mathbb{R}^n, \ \vec{a}^T \vec{x} = b \}$

$\boxed{H^-_{\vec{a}, b}} = \{ \vec{a}^T \vec{x} \le b \}$

$\boxed{H^+_{\vec{a}, b}} = \{ \vec{a}^T \vec{x} \ge b \}$

Theorem: Let $K \subseteq \mathbb{R}^n$ be convex and closed. Then through each boundary point of K, there is a support plane of K. If $K \neq \phi$ is bounded, then to each vector $\vec{a} \in \mathbb{R}^n \setminus \{\vec{0}\}$ there is a support plane to K with exterior normal vector $\vec{a}$.



Proof:

(1) Let $\vec{x} \in \text{bd } K$.

To begin with, assume K is bounded.

$\left[\, p(K, S) = \text{bd } K \,\right]$

$\exists\, \vec{y} \in S$, such that

~~$\vec{y} \in p(K, \vec{x})$~~

$\vec{x} = p(K, \vec{y})$

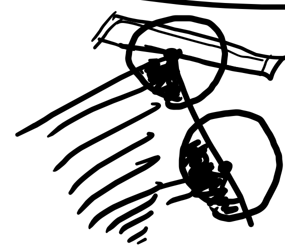
The hyperplane through $p(K, \vec{y}) = \vec{x}$ orthogonal to $\vec{y} - \vec{x}$ supports K at $\vec{x}$.

(2) Assume K is not bounded.

Through $\vec{x} \in \operatorname{bd} K$, there is a <u>supporting plane</u> H of $K \cap \overline{B(\vec{x}, 1)}$

ball of radius 1 centred at $\vec{x}$.

Let H^- be the corresponding supporting halfspace of $\boxed{K \cap \overline{B(\vec{x}, 1)}}$.

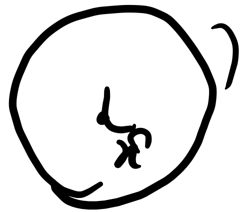


<u>Claim</u>: $K \subseteq H^-$.

<u>Proof of claim</u>: If $K \setminus H^-$ is nonempty, pick $\vec{z} \in K \setminus H^-$. (that is, $\vec{z} \in K \cap \operatorname{cmpl}(H^-)$)

$[\vec{z}, \vec{x}] \subset K$. But $[\vec{z}, \vec{x}] \cap \overline{B(\vec{x}, 1)}$

is not in H^-. Contradiction!!



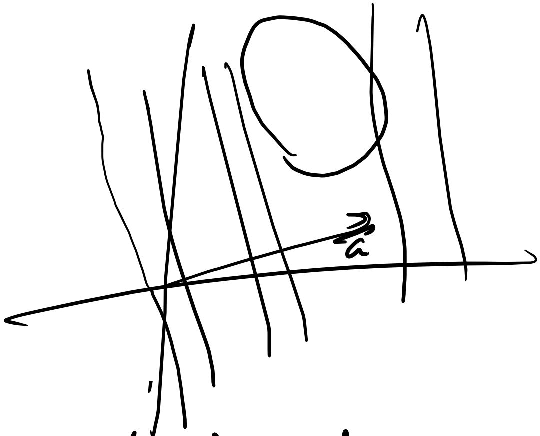
Hence H supports K.

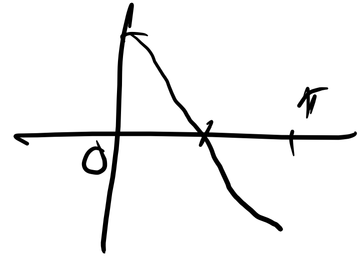
(iii) Let K be bounded and $\vec{a} \in \mathbb{R}^n \setminus \{0\}$.

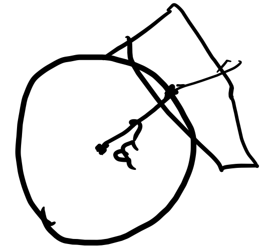
Since K is compact, there is an $\vec{x} \in K$

s.t. $\langle \vec{x}, \vec{a} \rangle = \sup \{ \langle \vec{y}, \vec{a} \rangle ; \vec{y} \in K \}$

$\langle \vec{a}, \cdot \rangle : K \to \mathbb{R}$







$\to L$

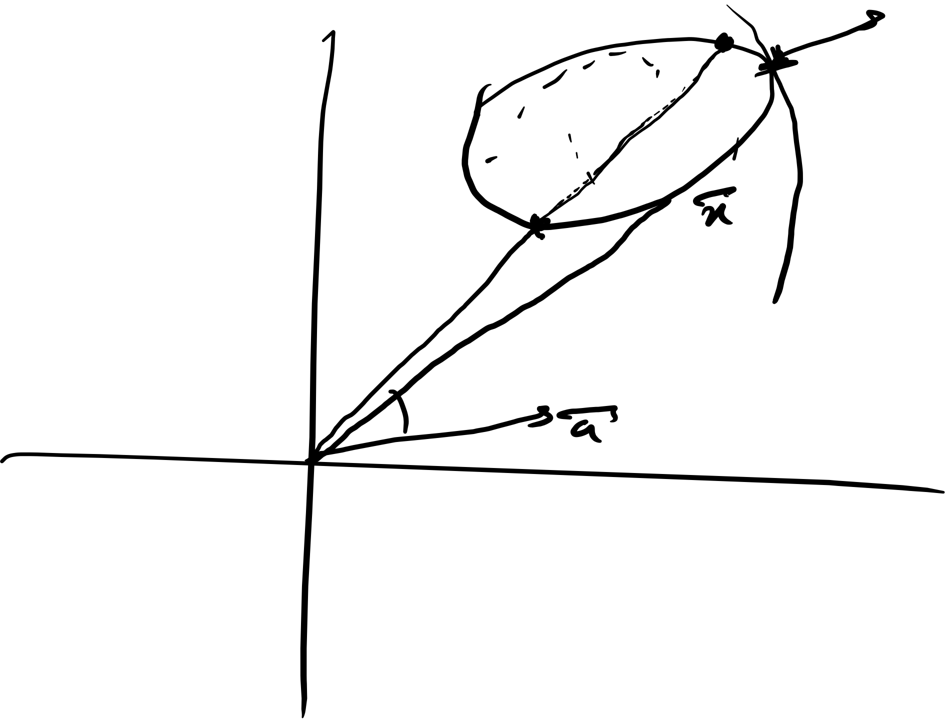
$$\underline{\{\vec{y} \in \mathbb{R}^n : \langle \vec{y}, \vec{a} \rangle = \boxed{\langle \vec{x}, \vec{a} \rangle}\}}$$

is supporting plane to K with exterior normal vector $\vec{a}$.

$\vec{z} \in K$, $\langle \vec{z}, \vec{a} \rangle \leq \langle \vec{x}, \vec{a} \rangle$

$= \langle \vec{y}, \vec{a} \rangle$

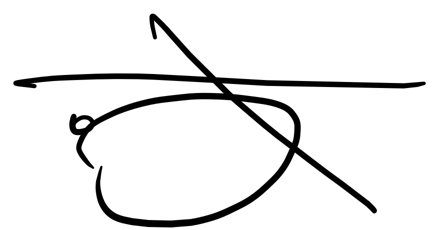
iff $\langle \vec{z} - \vec{y}, \vec{a} \rangle \leq 0$.



$$\langle \vec{x}, \vec{\xi} \rangle = \sup_{\vec{y} \in K} \langle \vec{y}, \vec{\xi} \rangle,$$

Exercise: Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set with $\text{int}\, K \neq \phi$ such that through each boundary point of K, there is a support plane of K. Show that K must be convex.

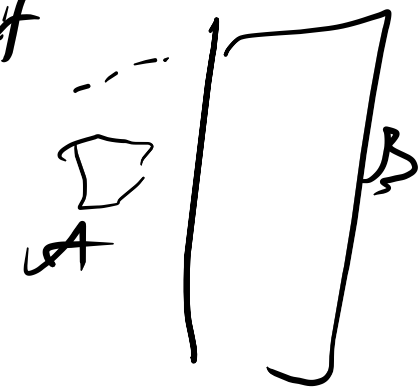
(We have discussed the support properties of convex sets?.)



Separation properties of convex sets:

Let $A, B \subseteq \mathbb{R}^n$ and $H_{\vec{a},b} \subseteq \mathbb{R}^n$ be a hyperplane. The hyperplane $H_{\vec{a},b}$ **separates** A and B if

$A \subseteq H^-_{\vec{a},b}$ and $B \subseteq H^+_{\vec{a},b}$.



The separation is said to be **proper** if A, B both do not lie in $H_{\vec{a},b}$.

The sets A, B are strictly separated by $H_{\vec{a},b}$ if $A \subseteq \operatorname{int}(H^-_{\vec{a},b})$, $B \subseteq \operatorname{int}(H^+_{\vec{a},b})$, or vice versa.

The sets A, B are strongly separated by $H_{\vec{a},b}$ if there is an $\varepsilon > 0$ such that $H_{\vec{a},b-\varepsilon}$ and $H_{\vec{a},b+\varepsilon}$ both separate A and B.

Separation of A and a point $\vec{x}$ means separation of A and $\{\vec{x}\}$.

Theorem:- Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $\vec{x} \in \mathbb{R}^n \setminus K$. Then K and $\vec{x}$ can be strongly separated.

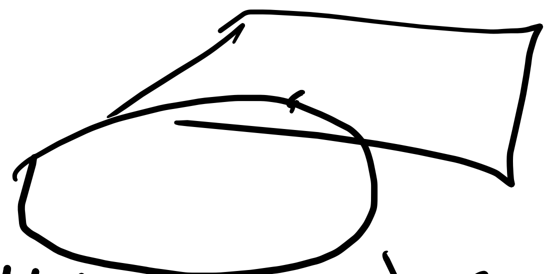
Proof :-

The hyperplane through $p(K, \vec{x})$ orthogonal to $u(K, \vec{x})$ supports K and hence separates $\vec{x}$ and K. The parallel hyperplane passing through $\frac{p(K, \vec{x}) + \vec{x}}{2}$ strongly separates K and $\vec{x}$.

REPRESENTATION THEOREM 1

(for closed convex set in $\mathbb{R}^n$)

Each nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspaces.



Pf:- If $\bar{x} \notin K$, there is a supporting halfspace which separates $\bar{x}$ and K.

$K \subseteq \bigcap$ supporting halfspace

$K^c \subseteq (\bigcap$ supporting halfspace$)^c \Rightarrow (K = \bigcap$ supporting halfspace$)$

Lemma: Let $A, B \subseteq \mathbb{R}^d$ be nonempty subsets. A and B can be separated (strongly separated, respectively) if and only if $\underbrace{A - B}_{\{\vec{x} - \vec{y} \,:\, \vec{x} \in A,\ \vec{y} \in B\}}$ and $\vec{0}$ can be separated (strongly separated, respectively).

($\Rightarrow$) Suppose $H_{\vec{u},b}$ strongly separates A, B, say,

$$\boxed{A \subseteq H^{-}_{\vec{u},\,\alpha-\varepsilon}}, \quad \boxed{B \subseteq H^{+}_{\vec{u},\,\alpha+\varepsilon}}$$

for some $\varepsilon > 0$.

Let $\vec{x} \in A - B$; then $\vec{x} = \vec{a} - \vec{b}$ for some $\vec{a} \in A$, $\vec{b} \in B$.

$$\langle \vec{a}, \vec{u} \rangle \leq \alpha - \varepsilon, \quad \langle \vec{b}, \vec{u} \rangle \geq \alpha + \varepsilon,$$

Thus, $\langle \vec{x}, \vec{u} \rangle \leq -2\varepsilon$, $\Rightarrow$ $A - B$ and $\vec{0}$ are strongly separated.

($\Leftarrow$) Suppose $A - B$ and $\vec{0}$ can be strongly separated $\Rightarrow \exists\, \vec{u} \in \mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ such that $\langle \vec{x}, \vec{u} \rangle \leq -2\varepsilon$ for all $\vec{x} \in A - B$. Let

$$\alpha := \sup \{ \langle \vec{a}, \vec{u} \rangle \mid \vec{a} \in A \}$$

$$\beta := \inf \{ \langle \vec{b}, \vec{u} \rangle \mid \vec{b} \in B \}.$$

$$\underline{\langle \vec{a}, \vec{u} \rangle} - \underline{\langle \vec{b}, \vec{u} \rangle} \leq -2\varepsilon$$

$$\Rightarrow \alpha - \beta \leq -2\varepsilon$$

$\beta - \alpha \geq 2\varepsilon_1$

Then, [$H_{\vec{u}, \frac{\alpha+\beta}{2}}$ strongly separates

A and B,]

[$\langle \vec{a}, \vec{u} \rangle \leq \frac{\alpha+\beta}{2}$]

$\langle \vec{b}, \vec{u} \rangle \geq \frac{\alpha+\beta}{2}$

$\alpha < \beta$

$\alpha \leq \frac{\alpha+\beta}{2} < \beta.$

$\langle \vec{a}, \vec{\varepsilon_1} \rangle \leq \alpha < \frac{\alpha+\beta}{2}$

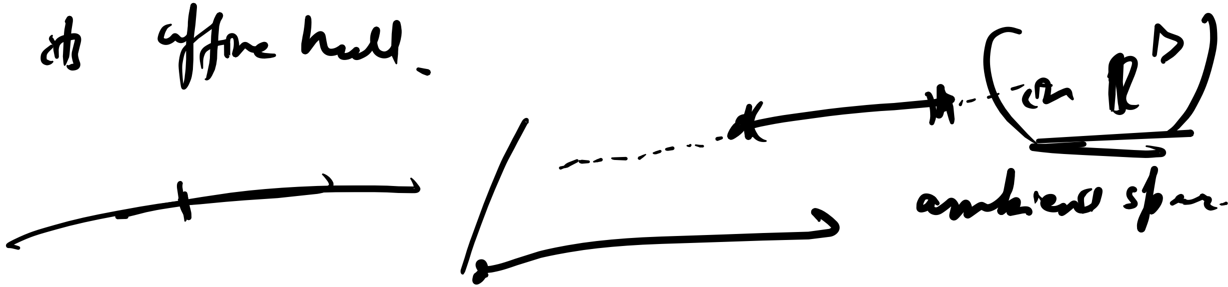
$\langle \vec{b}, \vec{\varepsilon_1} \rangle \geq \beta_1 > \frac{\alpha+\beta_1}{2}$

Exercise. Let $A, B \subseteq \mathbb{R}^n$ be nonempty convex sets such that A is compact and B is closed. If $A \cap B = \emptyset$, then A and B can be strongly separated.

EXTREMAL REPRESENTATIONS

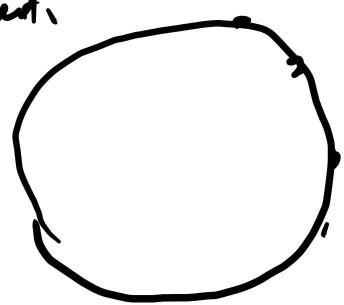
Defn: By $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{bd} S$, we mean the closure, interior, boundary of a subset S of $\mathbb{R}^n$.

The sets $\operatorname{relint}(S)$, $\operatorname{relbd}(S)$ are the relative interior, relative boundary of S, that is, the interior and boundary of S with respect to its affine hull.

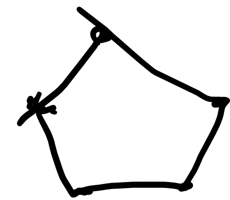


<u>Goal</u>: We want to represent a closed convex set as the convex hull of a smaller set; the smallest possible sets will be of particular interest.

<u>Lemma</u>:



Lemma, If $K \subseteq \mathbb{R}^n$ is closed convex set with [$K \neq \boxed{\text{conv}(\text{relbd } K)}$], then K is either an affine set or half-space of an affine set.



a $\boxed{(\bar{x} = \bar{d})}$ affine set

$\boxed{A\bar{x} \leq \bar{b}}$

Proof. Assume $\dim K = n$.

Let $\boxed{\bar{x}} \in \text{int}(K)$ with $\bar{x} \notin \text{conv}(\text{bd } K)$.

[$K \neq \boxed{\text{conv}(\text{bd } K)}$.

$\boxed{K = \text{int } K \cup \text{bd } K}$

Exercise. Show that relint $K \neq \emptyset$ for a closed convex set K in $\mathbb{R}^n$.

—

By the separation theorem, there is a closed halfspace such that $\vec{x} \in H^-$ and $\mathrm{conv}(\mathrm{bd}\, K) \subset H^+$.

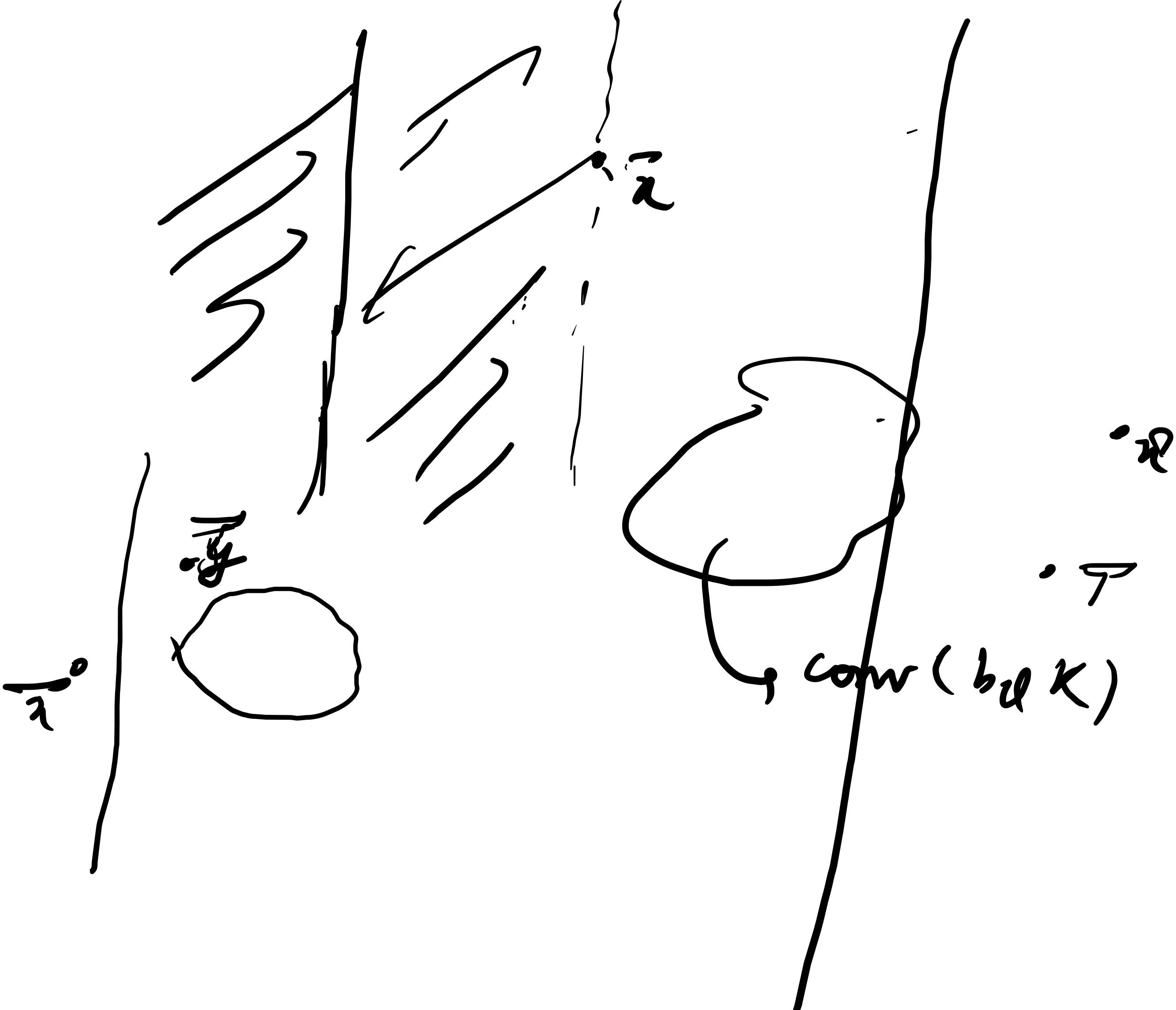
If $\vec{y} \in \mathrm{int}\, H^-$, then $\boxed{[\vec{x}, \vec{y}] \cap \mathrm{bd}\, K = \emptyset}$.

~~Then, $\vec{y} \in \mathrm{int}(K)$.~~

$\vec{x} \in K$. $\mathrm{conv}(\mathrm{bd}\, K) \subseteq K$.

$\vec{y} \in \mathrm{conv}(\mathrm{b}\ldots$

$[\vec{x}, \vec{y}] \subseteq K$.



x̄
conv(bd K)

Def^n: A set $A \subseteq \mathbb{R}^n$ is said to be **line-free** if A does not contain a line.

In particular, A does not contain a non-trivial affine set.

Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set. The **recession cone** of K

$$rec(K) = \{\vec{u} \in \mathbb{R}^n ; K + \vec{u} \subseteq K\}$$

$t k_1 + \vec{u}$

$+ (1-t) k_2$

$= k + t\vec{u}$

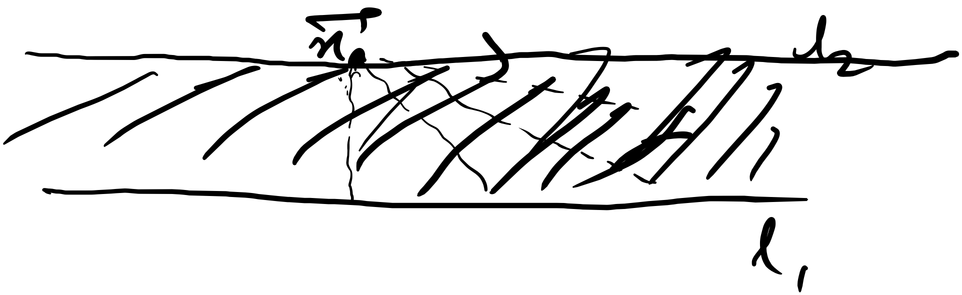
Exercise: $rec(K)$ is in fact a cone.

Lemma: Each closed convex set $K \subseteq \mathbb{R}^n$ can be represented in the form $K = \tilde{K} \oplus V$ where ~~is~~ V is a linear ~~span~~ subspace of $\mathbb{R}^n$ and $\tilde{K}$ is a line free closed convex set in a subspace complementing to V.

Proof : Assume K is not lineal-free.

$$V := \underline{\underline{\mathrm{rec}(K)}} \cap (-\underline{\underline{\mathrm{rec}(K)}}).$$

It is the linear subspace of $\mathbb{R}^n$ containing all vectors that are parallel to some line contained in K.



If $\vec{x} \in K$, $l_1 \subseteq \underline{K}$ then $l_2 \subset K$.

Let U be a linear subspace complementary to V.

Defn. $K_U := \underline{\underline{K}} \cap U$.

(Choose $x \in K \setminus V$. Then we may assume K_U is nonempty?

Note that $K_U + V \subseteq K$.

Let $\vec{x} \in K$.

Any line through $\vec{x}$ (l) which is completely contained in K must be parallel to V. Thus the line l must meet U at some point $\vec{y}$.

Then, $\vec{x} = \underline{\vec{y}} + \underline{(\vec{x}-\vec{y})}$, with $\vec{y} \in K_U$ and $\vec{x}-\vec{y} \in V$; hence $\vec{x} \in K_U + V$

K_v is closed, convex and loc. fin.