28/09/2021

Lecture - 3

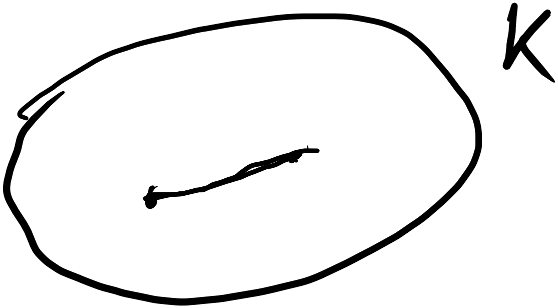
LP

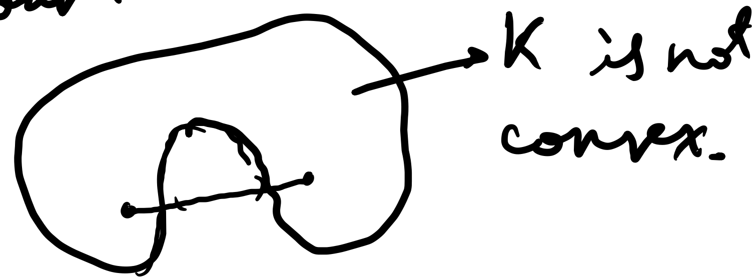
decision variables

min. $c_1 x_1 + \dots + c_n x_n$

subject to

$$\boxed{\begin{aligned} a_{11} x_1 + \dots + a_{1n} x_n &\le b_1 \\ &\vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n &\le b_m \end{aligned}}$$



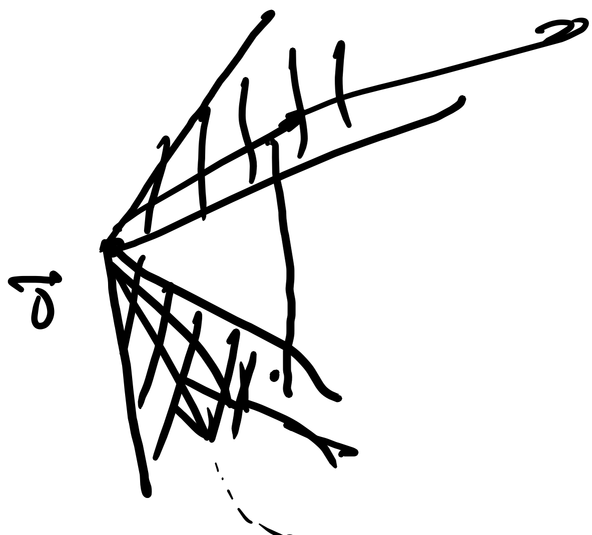


Cones in $\mathbb{R}^n$:

A set C in $\mathbb{R}^n$ is said to be a **cone** if for every $\vec{x} \in C$ and $\theta \geq 0$, we have $\theta \vec{x} \in C$.

A set C in $\mathbb{R}^n$ is said to be a convex cone if it is convex and a cone, which means that for any $\vec{x_1}, \vec{x_2} \in C$ and $\theta_1, \theta_2 \geq 0$, then

$$\underline{\theta_1 \vec{x_1} + \theta_2 \vec{x_2} \in C.}$$



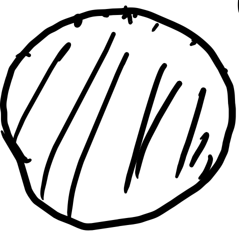
[C is closed under positive linear combinations]

Equivalently, if $\vec{x_1}, \ldots, \vec{x_k} \in C$, then

$\underline{\theta_1 \vec{x_1} + \ldots + \theta_k \vec{x_k}} \in C$ for $\theta_i \geq 0$.

conic combination
or positive combinations

Conic hull of $S \subseteq \mathbb{R}^n$



Let S be a subset of $\mathbb{R}^n$ (possibly infinite)
The conic hull of S is the smallest convex

cone in $\mathbb{R}^n$ containing S, in the following sense: If C' is any convex cone in $\mathbb{R}^n$ with $S \subseteq C'$, then $\text{conic}(S) \subseteq C'$

$\text{conic}(S)$ is $\{\underline{\theta_1 \vec{x_1} + \dots + \theta_k \vec{x_k}} : \vec{x_i} \in S, \theta_i \geq 0, i=1,\dots,k\}$

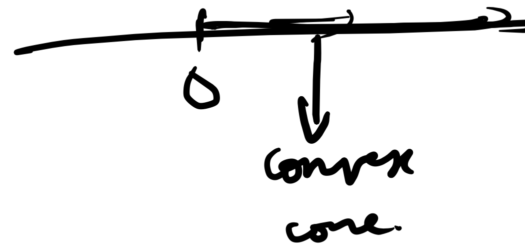
set of all conic combinations of points in S.

Generalized Inequalities

$\boxed{x \succeq y}$

$\boxed{x - y} \succeq 0.$

A cone $K \subseteq \mathbb{R}^n$ is called a <u>proper cone</u> if:



(1) K is convex

(2) K is closed

(3) K is solid. (nonempty interior)

(4) K is pointed, which means it contains only rays from the origin but no lines.

A proper cone defines a partial ordering $\precsim_K$ on $\mathbb{R}^n$; $\vec{x} \precsim_K \vec{y}$ if

$\vec{y} - \vec{x} \in K$ $\longrightarrow$ defines the notion of positivity.

(Exercise: Prove that $\precsim_K$ is a partial ordering.)

If $\vec{x} \precsim_K \vec{y}$, then

(1) $\vec{x} + \vec{z} \precsim_K \vec{y} + \vec{z}$ for any $\vec{z} \in \mathbb{R}^n$.

(2) $\alpha \vec{x} \precsim_K \alpha \vec{y}$, for any $\alpha \geq 0$.

(3) if $\vec{x}_i \precsim_K \vec{y}_i$ for $i = 1, 2, \ldots$ and $\vec{x}_i \to \vec{x}$, $\vec{y}_i \to \vec{y}$ then $\vec{x} \precsim_K \vec{y}$.

Examples:

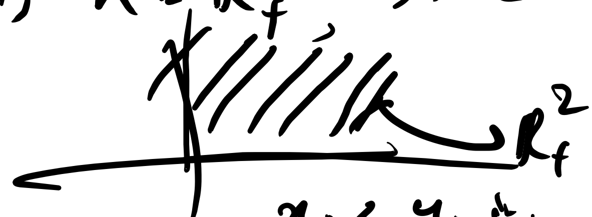
1) $K = \mathbb{R}_+ = [0, \infty)$ in $\mathbb{R}$.

$\preceq_K$ is the usual ordering $\leq$ on $\mathbb{R}$.

(ordinary inequalities in $\mathbb{R}$ are a special case of generalized inequalities.)

2) (Non-negative orthant, and componentwise inequality)

The non-negative orthant, $K = \mathbb{R}^n_+$, is a proper cone in $\mathbb{R}^n$.



$\preceq_K$ corresponds to componentwise inequality: $\vec{x} \preceq_{\mathbb{R}^n_+} \vec{y}$ means $x_i \leq y_i$, $i = 1, \dots, n$

Relevance to LP

decision space: $\boxed{\begin{array}{c} A\vec{x} \preceq \vec{b} \\ C\vec{x} = \vec{d} \end{array}}$

set of all $\vec{x}$ satisfying constraints

$$\boxed{\begin{array}{l} x_1 + x_2 \le 1 \\ x_1 + x_2 \le 1 \\ -x_1 - x_2 \le -1 \end{array}}$$

convex set

affine set

convex con.

$\vec{x}_1, \vec{x}_2 \in$ decision space

$\vec{b} - A\vec{x}_1 \in \mathbb{R}_+^n$

$\vec{b} - A\vec{x}_2 \in \mathbb{R}_+^n$

$$\left[\begin{array}{l} A\vec{x}_1 \preceq \vec{b} \\ A\vec{x}_2 \preceq \vec{b} \end{array}\right]$$

$$\left[\begin{array}{l} C\vec{x}_1 = \vec{d} \\ C\vec{x}_2 = \vec{d} \end{array}\right]$$

$$\theta(\vec{b} - A\vec{x}_1) + (1-\theta)(\vec{b} - A\vec{x}_2) \in \mathbb{R}_+^n$$

$$\Rightarrow \boxed{\vec{b}} - (\theta A\vec{x}_1 + (1-\theta)A\vec{x}_2) \in \mathbb{R}_+^n$$

$$\vec{b} - \left(A(\theta\vec{x}_1 + (1-\theta)\vec{x}_2)\right) \in \mathbb{R}_+^n$$

$$\Rightarrow$$

$\theta \qquad 1-\theta$

$$\frac{\theta_1}{\theta_1+\theta_2} \qquad \frac{\theta_2}{\theta_1+\theta_2}$$

$$\vec{b} \geq A(\theta\vec{x}_1 + (1-\theta)\vec{x}_2)$$

$$C(\theta\vec{x}_1 + (1-\theta)\vec{x}_2) = \vec{d}$$

$$\theta \cdot C\vec{x}_1 + (1-\theta)C\vec{x}_2$$

$$\theta\vec{d} + (1-\theta)\vec{d} = \vec{d}$$

$\vec{x}_1, \vec{x}_2 \in$ dec. space

$\mathrm{conv}(\vec{x}_1, \vec{x}_2) \in$ dec. space

Goal 1: To understand the structure of the decision space.

Goal 2: To make a computer understand the structure of the decision space.

Notation and definitions:

Let $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$.

(1) $\lambda_1 \vec{x}_1 + \dots + \lambda_k \vec{x}_k$ for $\lambda_i \in \mathbb{R}$ is a linear combination of $\vec{x}_1, \dots, \vec{x}_k$.

(2) $\lambda_1 \vec{x_1} + \dots + \lambda_k \vec{x_k}$ for $\lambda_i \in \mathbb{R}$

~~s.t.~~ $(\lambda_1 + \dots + \lambda_k = 1)$ is an affine combination

of $\vec{x_1}, \dots, \vec{x_k}$.

(3) $\lambda_1 \vec{x_1} + \dots + \lambda_k \vec{x_k}$ for $\underline{\lambda_i \geq 0}$

$\underline{(\lambda_1 + \dots + \lambda_k = 1)}$ is a convex combination

of $\vec{x_1}, \dots, \vec{x_k}$.

(4) $\lambda_1 \vec{x_1} + \dots + \lambda_k \vec{x_k}$ for $\lambda_i \geq 0$ is a

positive combination of

$\vec{x_1}, \dots, \vec{x_k}$.

Let S be a set in $\mathbb{R}^n$,

(1) <u>linear hull</u> of S is the set of <u>linear combinations</u> of S.

(S <u>linearly generates</u> $\mathrm{lin}(S)$) $\rightarrow$ $\mathrm{span}(S)$.

(2) <u>affine hull</u> of S is the set of <u>affine combinations</u> of S.

(S <u>"affinely" generates</u> $\mathrm{aff}(S)$).

(3) convex hull of S is the set of convex combinations of points in S.

(S "convexly" generates $\mathrm{conv}(S)$).

(4) positive hull of S is the set of positive combinations of points in S.

(S positively generates $\mathrm{pos}(S)$)

$$\mathrm{conv}(S) = \mathrm{aff}(S) \cap \mathrm{pos}(S)$$

We can ask the reverse question.

(1) Let V be a linear subspace of $\mathbb{R}^n$.
What is "smallest" set S in $\mathbb{R}^n$ s.t.
$\mathrm{lin}(S)$ (or $\mathrm{span}(S)$) $= V$?

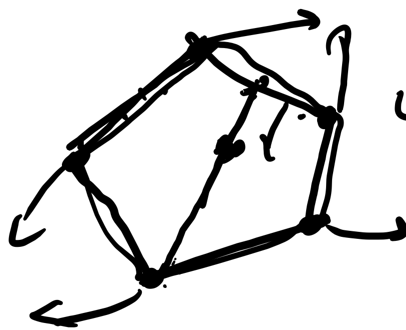
(basis)

(2) Let A be an affine set in $\mathbb{R}^n$.
What is "smallest" set S in $\mathbb{R}^n$ s.t.

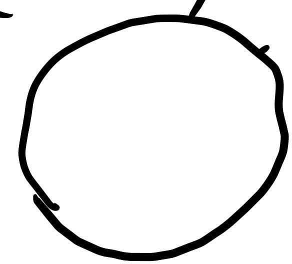
$\mathrm{aff}(S) = A$? (something like a basis. "affine basis").

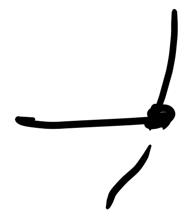
(3) Let C be a closed convex set in $\mathbb{R}^n$. What is the "smallest" set S in $\mathbb{R}^n$ such that

$$\mathrm{conv}(S) = C \, ?$$



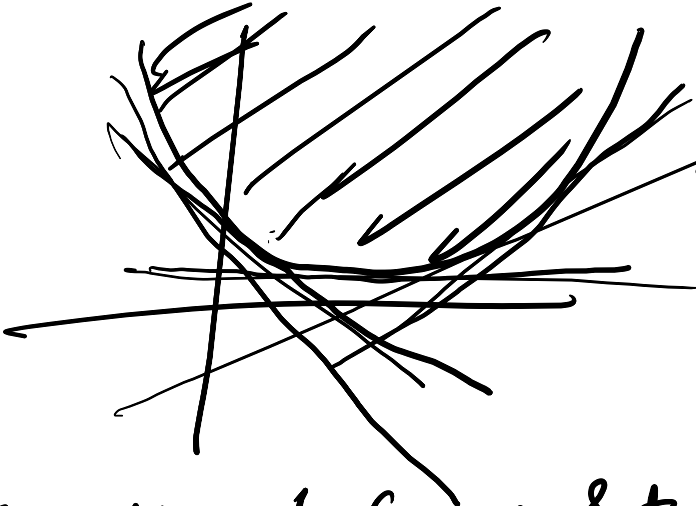
conv(vertices) = pentagon





~~bounded~~

compact convex sets



$\mathrm{epi}(f)$ is also conv

$$(a_1, \dots, a_n)^T \underline{x} = a_1 \underline{x_1} + \dots + a_n x_n = b$$

Examples of Convex Sets

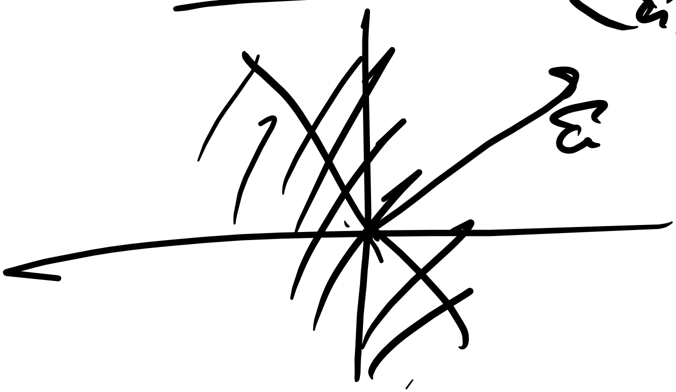
(1) Hyperplane and Halfspaces.

A hyperplane is a set of the form $H_{a,b} := \{\vec{x} \in \mathbb{R}^n : \vec{a}^T \vec{x} = \vec{b}\}$ where $\vec{a} \neq \vec{0}$.

~~If~~ $b = 0$.

$\vec{a}^T \vec{x} = \vec{0}$

$\langle \vec{a}, \vec{x} \rangle$



$H_{\vec{a},b}$ has dimension $n-1$.

(solution of ~~a~~ one nontrivial linear equality)

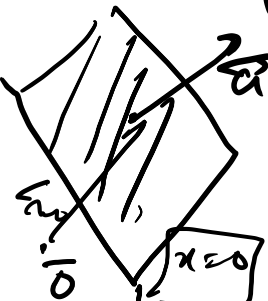
Geometrically, $H_{\vec{a},b}$ has normal vector $\vec{a}$.

$b \in \mathbb{R}$ determines the offset from the origin.

Let $\vec{x}_0$ be such that $\vec{a}^T\vec{x}_0 = b$.

(such an $\vec{x}_0$ exists?)

$\vec{a}^T\vec{x}_0 \cdot b = \vec{a}^T\vec{x}_0$



$$H_{\vec{a},b} = \{\vec{x} \in \mathbb{R}^n : \vec{a}^T(\vec{x} - \vec{x}_0) = 0\}$$

$\langle \vec{a}, \vec{x} - \vec{x}_0 \rangle$

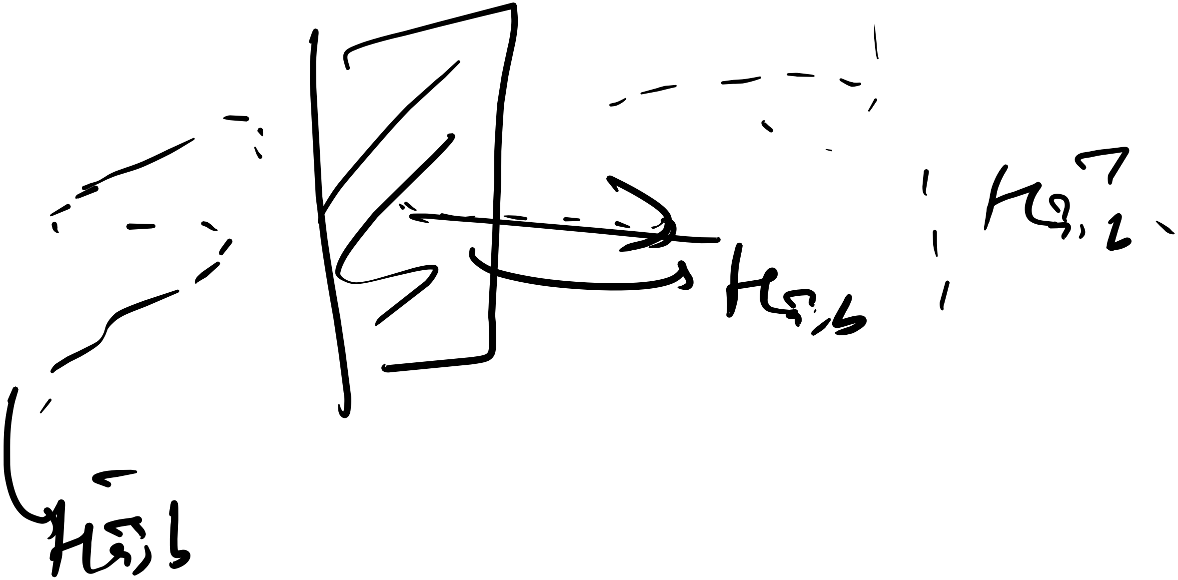
Then, $\vec{x} - \vec{x}_0 \perp \vec{a}$, $\forall \vec{x} \in H_{\vec{a},b}$.

Halfspace: A (closed) half-space in $\mathbb{R}^n$ is a set of the form

$$H^-_{\vec{a},b} := \{\vec{x} \in \mathbb{R}^n : \vec{a}^T\vec{x} \le b\}$$

$$H^+_{\vec{a},b} := \{\vec{x} \in \mathbb{R}^n : \vec{a}^T\vec{x} \ge b\}$$

$\vec{a} \neq 0$, $b \in \mathbb{R}$.



* Half spaces are convex but not affine.
(solution set of <u>one</u> nontrivial linear inequality)

$$H_{\vec{a},b} = H^{+}_{\vec{a},b} \cap H^{-}_{\vec{a},b}.$$

The boundary of $H^{+}_{\vec{a},b}$ and also of $H^{-}_{\vec{a},b}$ is $H_{\vec{a},b}$.

$\operatorname{int}(H^{-}_{\vec{a},b})$ is called an <u>open</u> half-space.

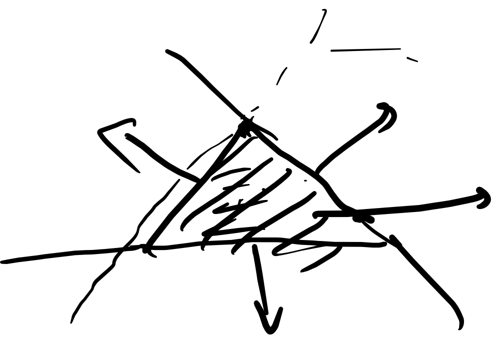
* In a certain sense, <u>hyperplanes</u> and <u>half-spaces</u> are building blocks of our <u>decision space</u>.

e) <u>Polyhedra</u>

A <u>polyhedron</u> is defined as the solution set of a finite number of linear equalities and inequalities.

$$P = \{ \vec{x} \in \mathbb{R}^n : \vec{a_j}^T \vec{x} \le b_j, \ j=1,\dots,m$$
$$\vec{c_j}^T \vec{x} = d_j, \ j=1,\dots,p \}$$

$$\vec{a_j} \neq \vec{0}, \quad \vec{c_j} \neq 0$$



polyhedron described by 3 linear inequalities.

$$P = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} \preceq \vec{b},\ C\vec{x} = \vec{d},$$
$$\vec{b} \in \mathbb{R}^m,\ \vec{d} \in \mathbb{R}^p \}$$

$A \to m \times n$ matrix

$C \to p \times n$ matrix

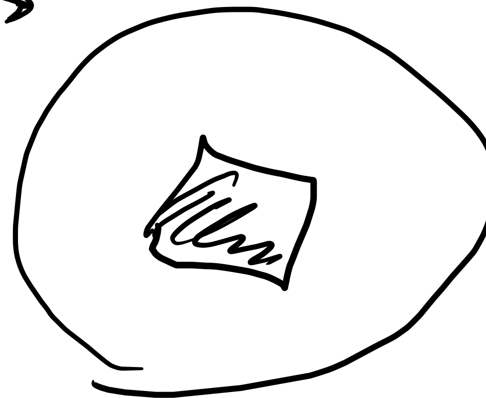
* A polyhedron is the intersection of a finite number of halfspaces and hyperplanes.
* (Affine sets) are all polyhedra.
* Polyhedra are closed convex sets.

$\vec{a}^T \vec{x} = b$

$\vec{x}_i \to \vec{x}$

$\vec{a}^T \vec{x}_i = b$

Defn: A bounded polyhedron is said to be a polytope.



Relevance to LP:

The decision space of an LP is a polyhedron. If the decision space is a polytope, existence of a minimum of the objective function is guaranteed.

$x_1 + x_2 \geq 1$

$x_1 + x_2 \leq$

Remark. As a general principle, we want to minimize convex objective function on a convex decision space. (effective algorithms to solve such problems).

[So why not understand general convex sets on $\mathbb{R}^n$?]

Defn: $\vec{x_1}, \dots, \vec{x_k} \in \mathbb{R}^n$ are said to be affinely independent if none of them is an affine combination of the others.

In other words, if $\sum_{i=1}^{k} \lambda_i \vec{x_i} = \vec{0}$ with $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{k} \lambda_i = 0$, then $\lambda_1 = \dots = \lambda_k = 0$.

$\vec{x_2} - \vec{x_1}$

$\vec{x_3} - \vec{x_1}$

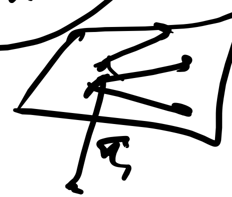
$\vdots$

$\vec{x_k} - \vec{x_1}$

are linearly independent

$$1 \cdot \vec{x_1} = \lambda_2 \vec{x_2} + \dots + \lambda_k \vec{x_k}, \quad \lambda_2 + \dots + \lambda_k = 1$$

$$\underline{1} \cdot \vec{x_1} + (\underline{-\lambda_2})\vec{x_2} \dots + (\underline{-\lambda_k})\vec{x_k} = \vec{0}$$



Theorem (Caratheodory's theorem)

If $S \subseteq \mathbb{R}^n$ and $\vec{x} \in \mathrm{conv}(S)$, then $\vec{x}$ is a convex combination of affinely independent points of S.

(In particular, $\vec{x}$ is a convex combination of $n+1$ or fewer points of S.)

Proof: $\vec{x} = \sum_{i=1}^{k} \lambda_i \vec{x}_i$, $\vec{x}_i \in S$, $\sum_{i=1}^{k} \lambda_i = 1$. $\boxed{\lambda_i > 0}$

Let k be the minimal such number.

Assume $\vec{x}_1, \dots, \vec{x}_k$ are affinely dependent.

$\exists\, \alpha_1, \dots, \alpha_k \in \mathbb{R}$ (not all 0)

and $\boxed{\alpha_1 + \dots + \alpha_k = 0}$ s.t.

$$\boxed{\sum_{i=1}^{k} \alpha_i \vec{x}_i = \vec{0}.}$$

Choose m such that $\boxed{\alpha_m > 0,}$ and

$\frac{\lambda_m}{\alpha_m}$ is minimum possible from those.

$$\boxed{\vec{x} = \sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) \vec{x}_i}$$

$$\lambda_i \geq \frac{\lambda_m}{\alpha_m} \alpha_i \qquad \frac{\lambda_i}{\alpha_i} \geq \frac{\lambda_m}{\alpha_m}$$

if $\alpha_i \neq 0$

$$\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \geq 0$$

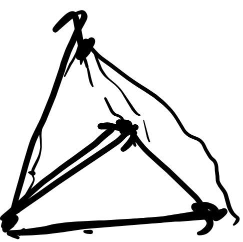
$$\sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) = 1 - \frac{\lambda_m}{\alpha_m} \left(\sum_{i=1}^{k} \alpha_i \right)$$

$= 1$

All $\lambda_m - \frac{\lambda_m}{\alpha_m} \cdot \alpha_m = 0$, contradicting the minimality of k.

∎

Defn: A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^n$ (k represents dimension of the simplex). These points are the vertices of the simplex.



$\vec{0}$

2-simplex.

$\vec{x}_1, \vec{x}_2, \vec{x}_3$ are affinely independent.

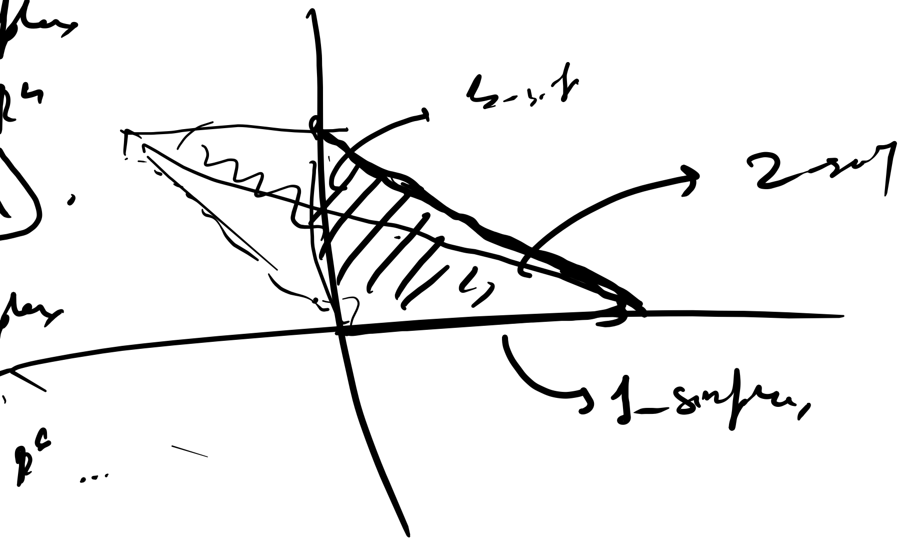
triangle → 2-simplex

tetrahedron → 3-simplex

k-simplex in $\mathbb{R}^n$

$\boxed{k \le n}$.

5-simplex in $\mathbb{R}^5$ or $\mathbb{R}^6$ ...



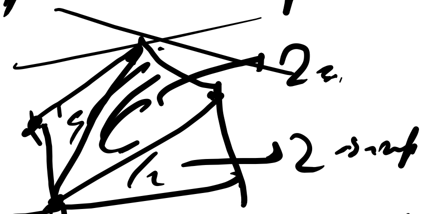
$$\boxed{\begin{aligned} & x_1 + \dots + x_n \le 1 \\ & x_1 \ge 0 \\ & \vdots \\ & x_n \ge 0 \end{aligned}}$$

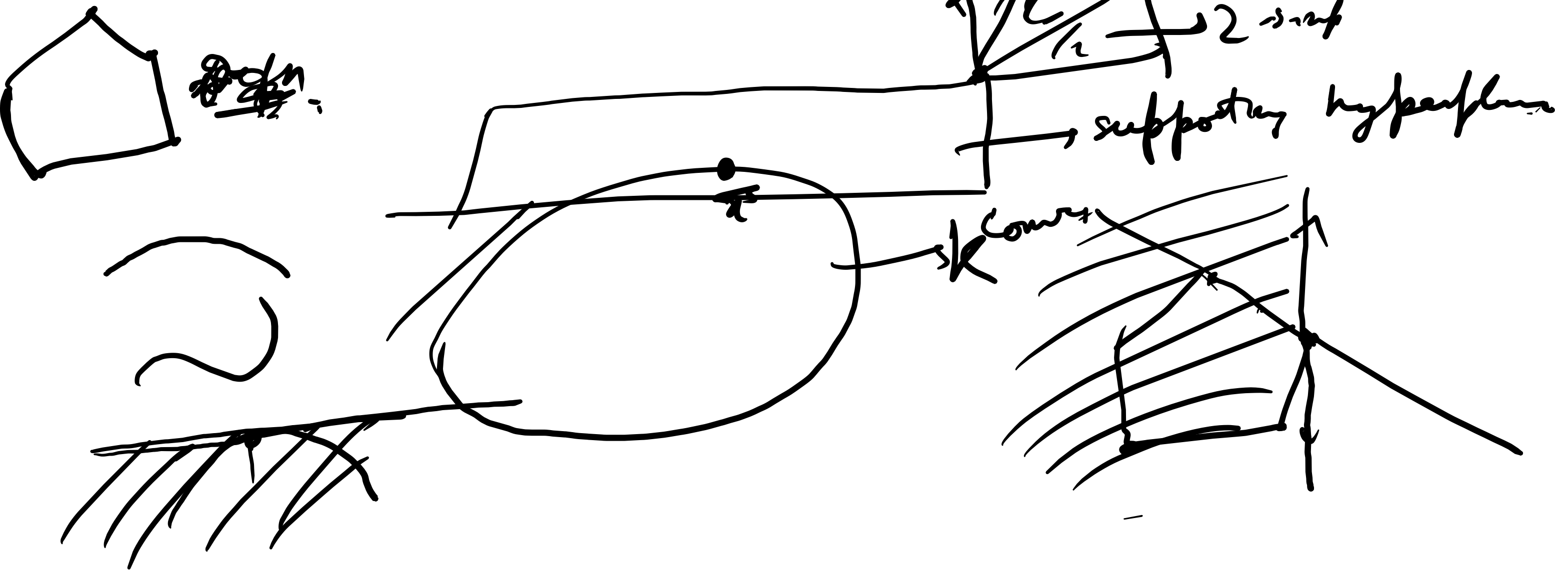
$x_1 + x_2 \le 1$

$x_1 \ge 0$

$x_2 \ge 0$

* Caratheodory's theorem states that conv(S) is the union of all simplices with vertices on S





Two main representation theorems for closed convex sets

REPRESENTATION THEOREM 1:

Each nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspaces.

REPRESENTATION THEOREM 2 (Extremal representation).

Each bounded closed (compact) convex set in $\mathbb{R}^n$ is the convex hull of its extreme points. (analogy with basis