23/09/2021 Lecture 2

$\vec{x} \to$ column vector in $\mathbb{R}^n$

Linear Program:

minimize $f_0(\vec{x}) = \vec{c}^T \vec{x}$

subject to $f_i(\vec{x}) \leq b_i$, $i = 1, \ldots, m$

f_i's are linear maps from $\mathbb{R}^n$ to $\mathbb{R}$.
↳ constraint function

More concretely,

$x_1, \dots, x_n \longrightarrow$ decision variables

minimize $\quad c_1 x_1 + \dots + c_n x_n \quad \left(= \vec{c}^T \vec{x} \right)$

subject to

$$\boxed{\begin{aligned} a_{11} x_1 + \dots + a_{1n} x_n &\leq b_1 \\ a_{21} x_1 + \dots + a_{2n} x_n &\leq b_2 \\ &\vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n &\leq b_m \end{aligned}}$$

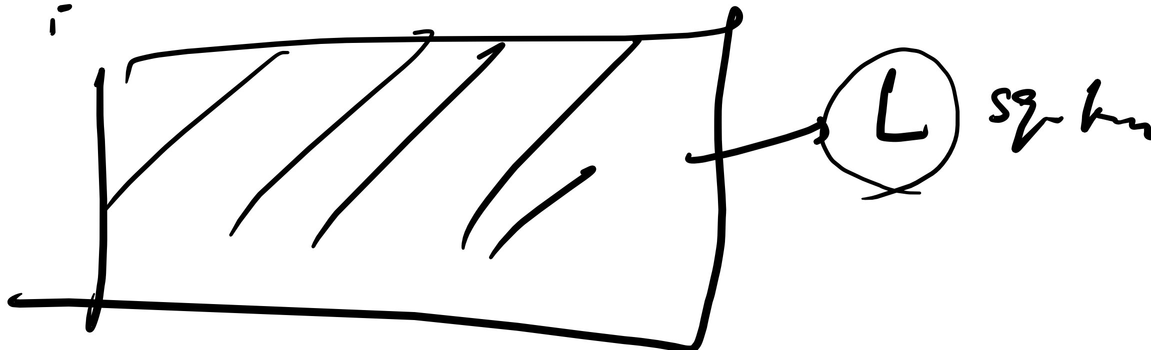
$\boxed{x_1 + x_2 = 1}$

$x_1 + x_2 \leq 1$

$(-x_1) + (-x_2) \leq -1$

Examples of LP:

Example 1:



Amount of fertilizer → F kg.

" " pesticide → P kg.

	Wheat	Maize
Fertilizer req. (per sq. km)	F_W	F_M
Pesticide req. (per sq. km)	P_W	P_M
selling price (per sq. km.)	$\underline{\underline{S_W}}$	$\underline{\underline{S_M}}$

Area of land allocated to $W \rightarrow x_W$

$M \rightarrow x_M$

" "

maximize: $\boxed{s_W x_W + s_M x_M}$ $\leftarrow$ $-\alpha$

subject to: $x_W + x_M \le L$ (total land area)

$F_W x_W + F_M x_M \le F$ (total amount of fertilizer)

$P_W x_W + F_{\cancel{M}} x_M \le P$ (" " of pesticide)

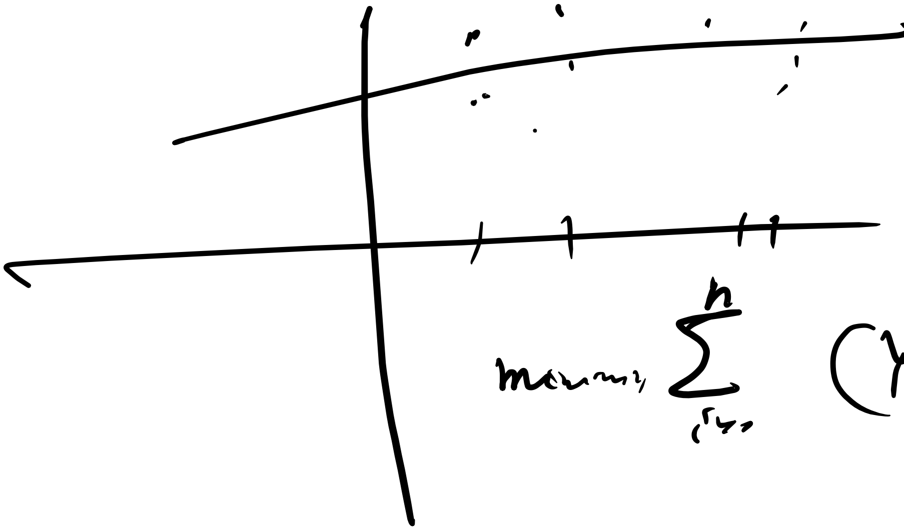
$x_W \ge 0, \quad x_M \ge 0.$

Example 2: (Linear regression)

$(X_1, Y_1), \dots, (X_n, Y_n)$

$X \rightarrow$ predictor variable, $Y \rightarrow$ response variable

$$Y = \boxed{\beta_0} + \boxed{\beta_1} X + \varepsilon$$



Find best fitting line to the dataset

$$\text{minimize} \sum_{i=1}^{n} (Y - \beta_0 - \beta_1 X)^2$$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \qquad \begin{bmatrix} \underline{\underline{\beta_0}} + \underline{\underline{\beta_1}} X_1 \\ \vdots \\ \underline{\underline{\beta_0}} + \underline{\underline{\beta_1}} X_n \end{bmatrix}$$

measure of $\underline{\text{closeness}}$ of the above two vectors.

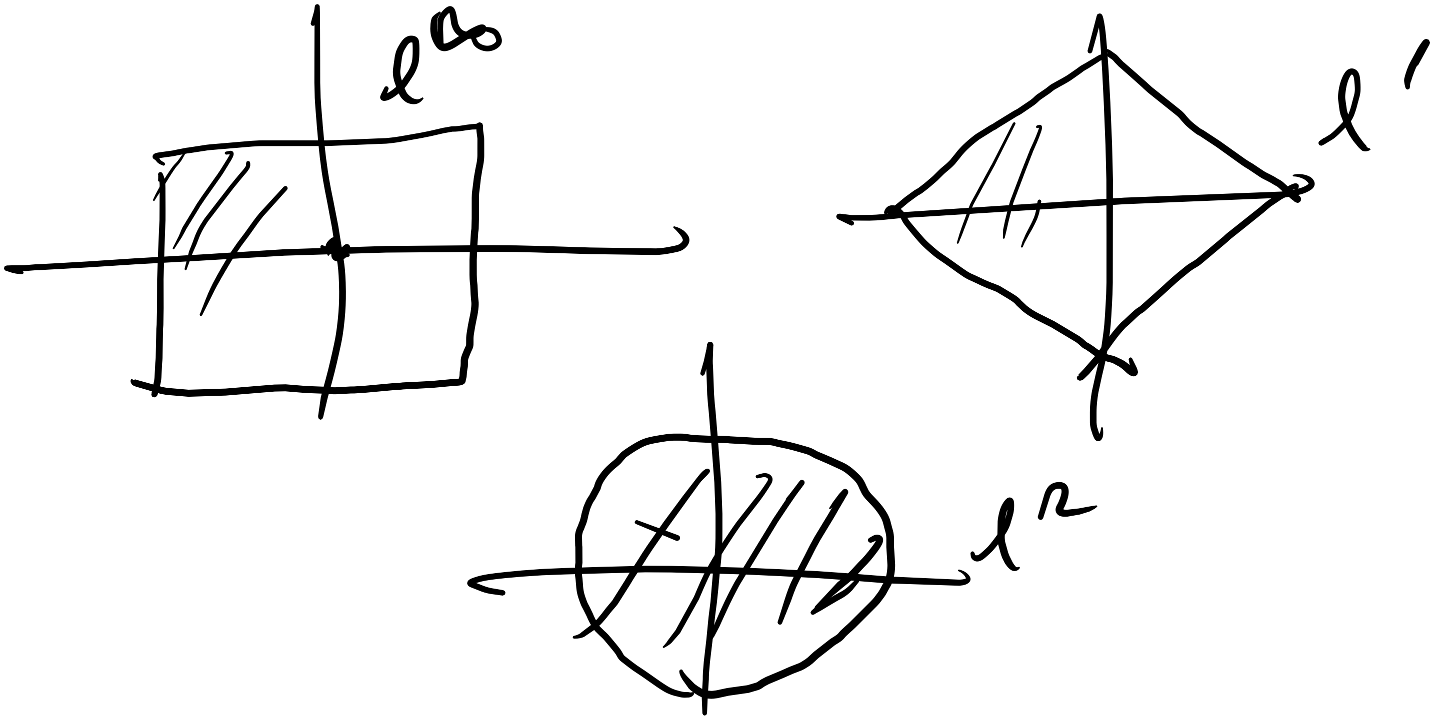
l^1-norm: $\cancel{\sum_{i=1}^{n} |x}$ $\quad \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$ $\quad \|\vec{x} - \vec{y}\| = \sum_{i=1}^{n} |x_i - y_i|$

l^2-norm: $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

l^p-norm: $\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

(Chebyshev approximation problem)

ℓ^∞ - norm: $\|\vec{x}\|_\infty = \max_{i=1,\dots,n} |x_i|$



$$\text{minimize} \left(\max_{i=1,\dots,n} |Y_i - \underline{\beta_0} - \underline{\beta_1} X_i| \right)$$

<u>Equivalent LP</u>:

$$\text{minimize} \quad t$$

subject to

$$\underline{\beta_0 + \beta_1} X_i + t \geq Y_i$$

$$\beta_0 + \beta_1 X_i - t \leq Y_i \qquad i=1,\dots,\underline{n}.$$

$$\underline{t \geq 0}$$

$$\beta_0, \beta_1, t \geq 0 \quad \Bigg| \quad Y_i - \beta_0 - \beta_1 X_i \leq t$$

$$\geq -t$$

$$\left[|Y_i - \beta_0 - \beta_1 X_i| \leq t \right]$$

GOAL: To understand the structure of the decision space in an LP.

$\boxed{\mathbb{R}^n}$

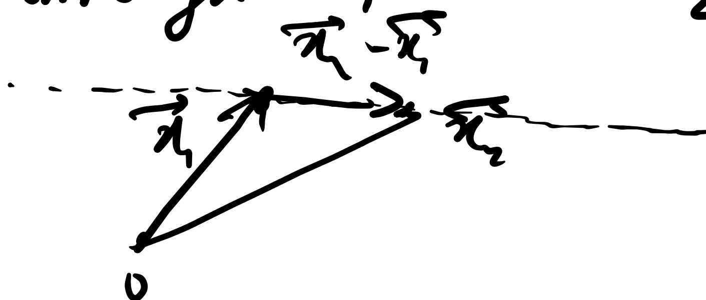
vector subspace $\rightarrow$ closed under $\boxed{\text{arbitrary}}$ linear combination.

Affine sets in $\mathbb{R}^n$, Convex sets in $\mathbb{R}^n$, Cones in $\mathbb{R}^n$.

AFFINE SETS

$\vec{x}_1 , \vec{x}_2 \in \mathbb{R}^n \quad (\vec{x}_1 \neq \vec{x}_2)$

Line passing through $\vec{x}_1$ and $\vec{x}_2$: given by



$$\vec{x}_1 + t(\vec{x}_2 - \vec{x}_1), \quad t \in \mathbb{R}$$

$$\Vert$$

$$\boxed{t\vec{x}_2 + (1-t)\vec{x}_1} \quad t \in \mathbb{R}.$$

Line segment joining $\vec{x_1}$ and $\vec{x_2}$:

$$\vec{x_1} + t(\vec{x_2} - \vec{x_1}), \quad t \in [0,1]$$

$$\shortparallel$$

$$\boxed{t\vec{x_2} + (1-t)\vec{x_1}}, \quad t \in [0,1]$$

<u>Affine set</u>: A set $A \subseteq \mathbb{R}^n$ is said to be <u>affine</u> if the unique line through two distinct points in A, lies in A.

In other words,

if $\vec{x_1}, \vec{x_2} \in A$, then $t\vec{x_1} + (1-t)\vec{x_2} \in A$

$\forall\ t \in \mathbb{R}$.

Equivalently, (prove by induction). affine linear combination of $\vec{x_1}, \dots, \vec{x_k}$

if $\vec{x_1}, \dots, \vec{x_k} \in A$, then $t_1\vec{x_1} + \dots + t_k\vec{x_k} \in A$

$\forall\ t_1, \dots, t_k \in \mathbb{R}$ s.t. $\boxed{t_1 + \dots + t_k = 1}$

<u>Remark</u>: If A is an affine set, and $\vec{x_0} \in A$,

then $V := A - \vec{x_0} = \{\vec{x} - \vec{x_0} : \vec{x} \in A\}$

is a vector subspace of $\mathbb{R}^n$ (Exercise).

In other words, every affine subset of $\mathbb{R}^n$ is a translation of a subspace of $\mathbb{R}^n$.

Example: (Solution set of a system of linear equations

$\mathcal{A} = \{ \vec{x} : A\vec{x} = \vec{b} \}$. where $A \to m \times n$ matrix, $\vec{b} \in \mathbb{R}^{m}$

(m linear equations

$A\vec{x}_1 = \vec{b}$, $A\vec{x}_2 = \vec{b}$

$\to$ particular solution + solution to homogeneous eqn

(translate) (subspace of $\mathbb{R}^n$

<u>Prop^n</u>: Every affine set in $\mathbb{R}^n$ can be expressed as the solution set of a system of linear equations.

Proof. A

$V = A - x_0$ is a subspace of $\mathbb{R}^n$.

* We will express V as the solution set of a system of homogeneous linear equations.

Let $w_1, \dots, w_r$ be an orthonormal basis of $V^\perp$

(that is $\langle w_i, w_j \rangle = \delta_{ij}$, $\langle w_j, v \rangle = 0 \quad \forall v \in V$).

$$\underbrace{[w_1 \mid w_2 \mid \cdots w_k]^T}_{A} v = 0$$

Mantra:

Affine set in $\mathbb{R}^n$ = translate of a subspace of $\mathbb{R}^n$ = solution set of system of linear equations in n variables

Relevance to LP

If the constraint functions defining the decision space were all equalities, the decision space would be an affine subset of $\mathbb{R}^n$.

Affine hull of $S \subseteq \mathbb{R}^n$

Let S be a subset of $\mathbb{R}^n$ (possibly infinite). The affine hull of S (denoted $\text{aff}(S)$) is the smallest affine set in $\mathbb{R}^n$ containing S, or the following sense. If A' is an affine set in $\mathbb{R}^n$ with $S \subseteq A'$, then

$\text{aff}(S) \subseteq A'$.

$$\left[\text{aff}(S) = \{ \theta_1 \vec{x}_1 + \cdots + \theta_k \vec{x}_k : \vec{x}_1, \ldots, \vec{x}_k \in S, \ \theta_1 + \cdots + \theta_k = 1, \ k \in \mathbb{N} \} \right]$$

set of all affine linear combinations of points in S.

CONVEX SETS

A set $C \subseteq \mathbb{R}^n$ is said to be **convex** if the line segment joining any two distinct points in C, also lies in C.

If $\vec{x}_1, \vec{x}_2 \in C$, then $t\vec{x}_1 + (1-t)\vec{x}_2 \in C$

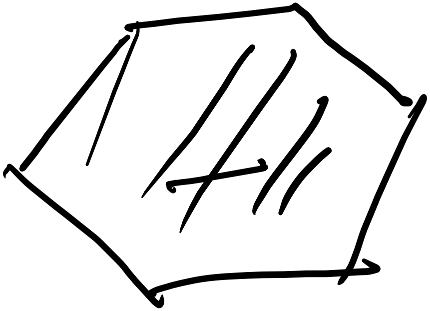
$$\forall \quad t \in [0,1]$$

Equivalently (prove by induction; exercise)

If $\vec{x}_1, \ldots, \vec{x}_n \in C$ then $\theta_1 \vec{x}_1 + \ldots + \theta_k \vec{x}_n \in C$

for $\boxed{\theta_i \geq 0}$ and $\theta_1 + \ldots + \theta_k = 1$.

ex All affine sets in $\mathbb{R}^n$ are convex.



convex

not convex

Convex hull of $S \subseteq \mathbb{R}^n$

Let S be a subset of $\mathbb{R}^n$ (possibly infinite).
The **convex hull** of S (denoted $\mathrm{conv}(S)$), is the smallest convex set in $\mathbb{R}^n$ containing S, in the following sense;

If C' is any convex set in $\mathbb{R}^n$,
and $S \subseteq C'$, then $\mathrm{conv}(S) \subseteq C'$.

$$\mathrm{conv}(S) = \{ \theta_1 \vec{x}_1 + \dots + \theta_k \vec{x}_k \mid \vec{x}_1, \dots, \vec{x}_k \in S, \; k \in \mathbb{N}, \; \theta_1 + \dots + \theta_k = 1, \; \theta_i \geq 0 \}$$

→ set of all convex combinations of points in S.

Relevance of convex sets in LP.