Optimization - B. Math, 6th Semester

Assignment 3 — Even Semester 2024-2025

Due date: February 10, 2025

Note: Each question is worth 10 points, and subparts are worth equal points. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only** after you have understood it. Please submit your assignment writeup in the beginning of class.

1. Let $A \subset \mathbb{R}^n$ be closed and convex. Show that $\text{ext}(A) \neq \varnothing$ if and only if A does not contain any line.

2. Let $K \subset \mathbb{R}^n$ be compact and convex.
   (a) If $n = 2$, show that $\text{ext}(K)$ is closed.
   (b) If $n \geq 3$, show by an example that $\text{ext}(K)$ need not be closed.

3. A real (n, n)-matrix $A = ((\alpha_{ij}))$ is called *doubly stochastic* if $\alpha_{ij} \geq 0$ and $\sum_{k=1}^n \alpha_{kj} = \sum_{k=1}^n \alpha_{ik} = 1$ for $i, j \in \{1, \ldots, n\}$. A doubly stochastic matrix with components in $\{0, 1\}$ is called a permutation matrix. Prove the following statements.
   (a) The set $K \subset \mathbb{R}^{n^2}$ of doubly stochastic matrices is compact and convex.
   (b) The extreme points of K are precisely the permutation matrices.

4. Let $A \subset \mathbb{R}^n$ be nonempty, closed and convex. Show that A is compact if and only if for each $\vec{u} \in S^{n-1}$ there is some $b \in \mathbb{R}$ such that $A \subset H^-_{\vec{a},b}$.

5. Prove that a polytope has finitely many extreme points. (A polyhedron is defined by finitely many linear equalities and inequalities. A polytope is a bounded polyhedron.)

Soumyashant Nayak: soumyashant@isibang.ac.in

6. (a) Suppose that Z is a random variable taking values in the set $\{0, 1, \ldots, K\}$, with probabilities $p_0, p_1, \ldots, p_K$, respectively. We are given the values of the first two moments $E[Z] = \sum_{k=0}^{K} kp_k$ and $E[Z^2] = \sum_{k=0}^{K} k^2 p_k$ of Z, and we would like to obtain upper and lower bounds on the value of the fourth moment $E[Z^4] = \sum_{k=0}^{K} k^4 p_k$ of Z. Provide a linear programming formulation of this problem.

   (b) Consider a set $\mathcal{P}$ described by linear inequality constraints, that is,

$$\mathcal{P} := \{\vec{x} \in \mathbb{R}^n | \vec{a_i}^T \vec{x} \leq b_i, i = 1, \ldots, m\}.$$

   A ball with center $\vec{y}$ and radius r is defined as the set of all points within (Euclidean) distance r from $\vec{y}$. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set $\mathcal{P}$. (The center of such a ball is called the Chebychev center of P.) Provide a linear programming formulation of this problem.

Soumyashant Nayak: soumyashant@isibang.ac.in