Optimization - B. Math, 6th Semester

Assignment 2 — Even Semester 2024-2025

Due date: February 05, 2025

Note: Each question is worth 10 points, and subparts are worth equal points. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only** after you have understood it. Please submit your assignment writeup in the beginning of class.

1. Let K be a nonempty convex set in $\mathbb{R}^n$.
   (a) Show that the closure of K is convex, and is equal to the closure of $\text{relint}(K)$. (The relative interior of K, $\text{relint}(K)$, is the interior of K relative to the affine hull of K.)
   (b) Let $A \subseteq \mathbb{R}^n$ be a nonempty convex set (not necessarily closed) and let $\vec{x} \in \mathbb{R}^n \backslash A$. Show that A and $\vec{x}$ can be separated.

2. (a) Show that the convex hull of a compact set in $\mathbb{R}^n$ is compact.
   (b) 1. Give an example of a closed set C in $\mathbb{R}^n$ (for some $n \in \mathbb{N}$) such that the convex hull of C is not closed. Provide a brief justification.
   2. Give an example of two closed convex sets in $\mathbb{R}^2$ that are disjoint but cannot be strictly separated. Provide a brief justification.

3. (a) *(Converse of supporting hyperplane theorem)*
   Suppose $K \subseteq \mathbb{R}^n$ is closed with nonempty interior, and has a supporting hyperplane at every point in its boundary. Show that K is convex.
   (b) *(A parametrization of the family of separating hyperplanes)*
   Suppose that C and D are disjoint subsets of $\mathbb{R}^n$. Consider the set of points $(\vec{a}, b) \in \mathbb{R}^{n+1}$ for which $\langle \vec{a}, \vec{x} \rangle \leq b$ for all $\vec{x} \in C$, and $\langle \vec{a}, \vec{x} \rangle \geq b$ for all $\vec{x} \in D$. (Note that $H_{\vec{a},b}$ is a supporting hyperplane for C and D.) Show that this set is a convex cone.

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4. The support function of a closed convex set $K \subseteq \mathbb{R}^n$ is defined as

$$S_K(\vec{y}) = \sup\{\langle \vec{y}, \vec{x} \rangle : \vec{x} \in K\} \leq \infty,$$

for every unit vector $\vec{y}$ in $\mathbb{R}^n$. (Let S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Note that S_K is a function from S^{n-1} to the extended real line $(-\infty, \infty]$.)

Give a geometric interpretation of the quantity $S_K(\vec{y}) - S_K(-\vec{y})$ for $\vec{y} \in S^{n-1}$. Suppose that K and L are closed convex sets in $\mathbb{R}^n$. Show that $K = L$ if and only if $S_K = S_L$.

5. Let A be an $m \times n$ matrix. Show that there exists an invertible $m \times m$ matrix P, and an invertible $n \times n$ matrix Q such that PAQ has the block form

$$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Conclude that row rank of A is equal to column rank of A.

6. (a) Use Gaussian elimination to find the inverse of the matrix A and solve $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 6 \\ 11 \\ 3 \end{bmatrix}$$

(b) Do elementary row operations alter the row space of a matrix? Do they alter the column space of a matrix? Provide justification or counterexample.

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