Optimization - B. Math, 6th Semester

Assignment 1 — Even Semester 2024-2025

Due date: January 27, 2025

Note: Each question is worth 10 points, and subparts are worth equal points. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only** after you have understood it. Please submit your assignment writeup in the beginning of class.

1. Verify Perron's theorem by computing the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 8 & 3 \\ 1 & 2 & 9 \end{pmatrix}$$

   Find the right-hand Perron vector $\mathbf{p}$ as well as the left-hand Perron vector $\mathbf{q}^T$.

2. (a) Prove that the Perron vector for an entry-wise positive matrix, $\mathbf{A}_{n\times n} > 0$, is unique.

   (b) Find the Perron root and the Perron vector for

$$\mathbf{A} = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$$

   where $\alpha + \beta = 1$ with $\alpha, \beta > 0$.

3. Suppose that $\mathbf{A}_{n\times n} > 0$ has $\rho(\mathbf{A}) = r$.

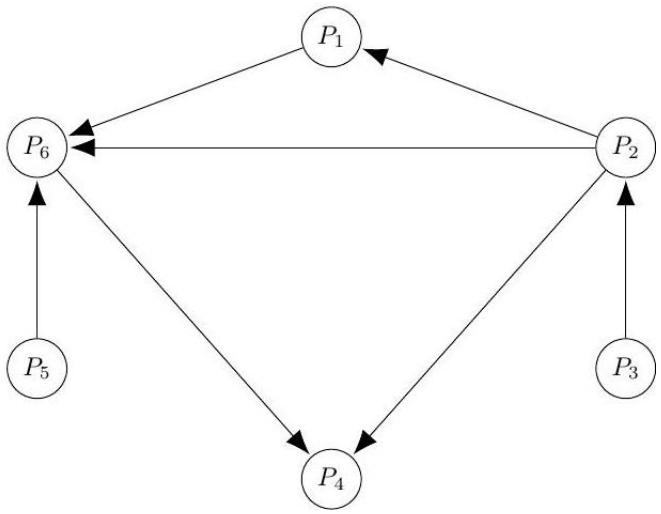
   (a) Prove that $\lim_{k\to\infty}(\mathbf{A}/r)^k$ exists and show that it is a non-zero idempotent matrix (which we denote by $\mathbf{G}$).

   (b) Prove that the range of $\mathbf{G}$ is the nullspace of $\mathbf{A} - r\mathbf{I}$ and its kernel is the range of $\mathbf{A} - r\mathbf{I}$.

Soumyashant Nayak: soumyashant@isibang.ac.in

4. To show the extent to which the hypothesis of positivity cannot be relaxed in Perron's theorem, construct examples of square matrices $\mathbf{A}$ such that $\mathbf{A} \geq \mathbf{0}$, but $\mathbf{A} \not> \mathbf{0}$ (i.e., $\mathbf{A}$ has at least one zero entry), with $r = \rho(\mathbf{A})$ that demonstrate the validity of the following statements. Different examples may be used for the different statements.
   (a) r can be 0.
   (b) $\operatorname{alg\,mult}_{\mathbf{A}}(r)$ can be greater than 1.
   (c) The r-eigenspace of $\mathbf{A}$ need not contain a positive eigenvector.
   (d) r need not be the only eigenvalue on the spectral circle.

5. Consider the mini-internet:



   (a) What is the hyperlink matrix of the above mini-internet?
   (b) Compute the Pagerank of the six webpages. First describe the setup of the problem (use damping factor of 0.15) and the strategy for computing page ranks. You may use a computer to find the values of the page ranks (approximately).

6. Why does the Google Pagerank produce more reasonable results than simply assigning a page a ranking in accordance with the number of pages that link to it?

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