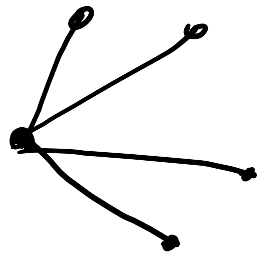
28/10/2021

Lecture - 11

F



1-dim

$\boxed{\text{k-dimensional faces}}$

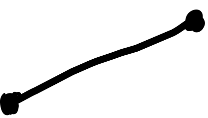
$$\left[A_1 \,|\, A_2 \,|\, \cdot \quad |\, A_n \right]$$

Updated cost - old cost

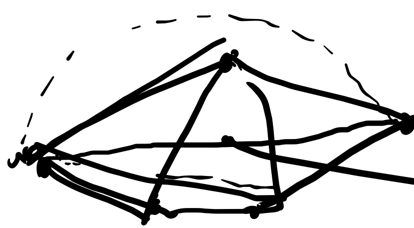
$$= \theta_0 \left[c_j - \sum_{i=1}^{m} c_{B(i)} x_{ij} \right]$$

want 'j' such that this is negative

A closed <u>k-dimensional face</u> of F is determined by $n-k$ linearly independent constraints and all points of that face are tight at these constraints.



1-dimensional face if v_1 and v_2 are neighbouring



not a 1-D face of

$n-m$ of the coordinates of a basic feasible solution are equal to 0.

$$I\vec{x} \geq \vec{0}$$

$$A\vec{x} = \vec{b}$$

$$\begin{array}{cc} & n \\ m \times n & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} & B(1) & B(m) \\ A & \Big[& \Big] \\ & A_{B(1)} & A_{B(m)} \end{array}$$

$$\vec{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\vec{b}$$

$$\vec{x}_{\mathcal{B}^c} = \vec{0}$$

$n-m$ linearly independent constraints binding at $\vec{x}$.

$n-1$ linearly ind

$\vec{v_1}, \vec{v_2}$ should have n linearly independent ~~constraint~~ tight constraint common.

$n-m$ of them are already common.

$c_B \to$ vector on $\mathbb{R}^m$ whose i^{th} component is $c_{B(i)}$.

$$\langle \vec{c}, \vec{x_B} \rangle = \sum_{i=1}^{m} c_{B(i)} x_{B(i)} = \langle \vec{c_B}, \vec{x_B} \rangle \quad \text{in } \mathbb{R}^m$$

$\vec{c}$ on $\mathbb{R}^n$

$\vec{x_B}$: bfs corresponding to $B = \{B(1), \dots, B(m)\}$.

$$Z_j = \sum_{i=1}^{m} x_{ij}\, c_{B(i)} = \left[(\vec{c}_B^{\,T}(X))_j\right] = (\vec{c}_B^{\,T} X)_j$$

$$\vec{c}\,' = \vec{c} - \vec{z} = \left[c_j - \sum_{i=1}^{m} x_{ij}\, c_{B(i)} \right]$$

modified cost

Theorem. If $\vec{c} - \vec{z} \geq 0$ at bfs $\vec{x_0}$, then $\vec{x_0}$ is optimal.

Proof. Let $\vec{y} \in F$. $\underline{\underline{A\vec{y} = \vec{b}}}$, $\vec{y} \geq 0$,

Let $B(1) \dots, B(m)$ be the current ordered basis. For all j, $\underline{\underline{c_j \geq z_j}}$. (by hypothesis).

So, $\boxed{c_j y_j \geq z_j y_j}$ for all j, $m\begin{bmatrix} A \end{bmatrix}$ $m \times n$ matrix.

$\Rightarrow \langle \vec{c}, \vec{y} \rangle \geq \langle \vec{z}, \vec{y} \rangle$

$= \boxed{\vec{z}^T \vec{y} = (\vec{c}_B^T \underline{\underline{B}}^{-1} A) \vec{y}}$ $\underbrace{\begin{bmatrix} A_{B1} | \dots | A_{Bm} \end{bmatrix}}_{B}$

$$X = K_s \cdots K_1 A \rightarrow [A_1 \cdots \mid A_2 \cdots$$

$$(X_{B(j)}) \qquad K := K_s \cdots K_1$$

$$K A_{B(i)} = e_i$$

$$K \underbrace{[A_{B(1)} \cdots \mid A_{B(m)}]}_{B} = [e_1 \mid \cdots \mid e_m]$$

$$= I$$

$$\Rightarrow \quad K = B^{-1} \qquad \boxed{X = B^{-1} A}$$

$$\langle \vec{c}, \vec{y} \rangle = \vec{c}_B^{\,T} B^{-1} A \vec{y}$$

$$= \vec{c}_B^{\,T} B^{-1} \vec{b}$$

Cost of $\vec{x}_0$ is

$$\langle \vec{c}, \vec{x}_0 \rangle = \boxed{\sum_{i=1}^{m} c_{B(i)} x_{i0}}$$

$$\langle \vec{c}, \vec{x}_0 \rangle = \langle \vec{c}_B, \vec{x}_B \rangle$$

$$= \vec{c}_B^{\,T} \begin{bmatrix} x_{10} \\ \vdots \\ \vdots \\ x_{m0} \end{bmatrix}$$

$$= \vec{c}_B^{\,T} B^{-1} \vec{b}$$

$$\begin{bmatrix} x_{10} \\ \vdots \\ x_{m0} \end{bmatrix}$$

$$A\vec{x} = A_B x_B + A_{B^c} x_{B^c}$$

$$= \vec{b}$$

$$\vec{0}$$

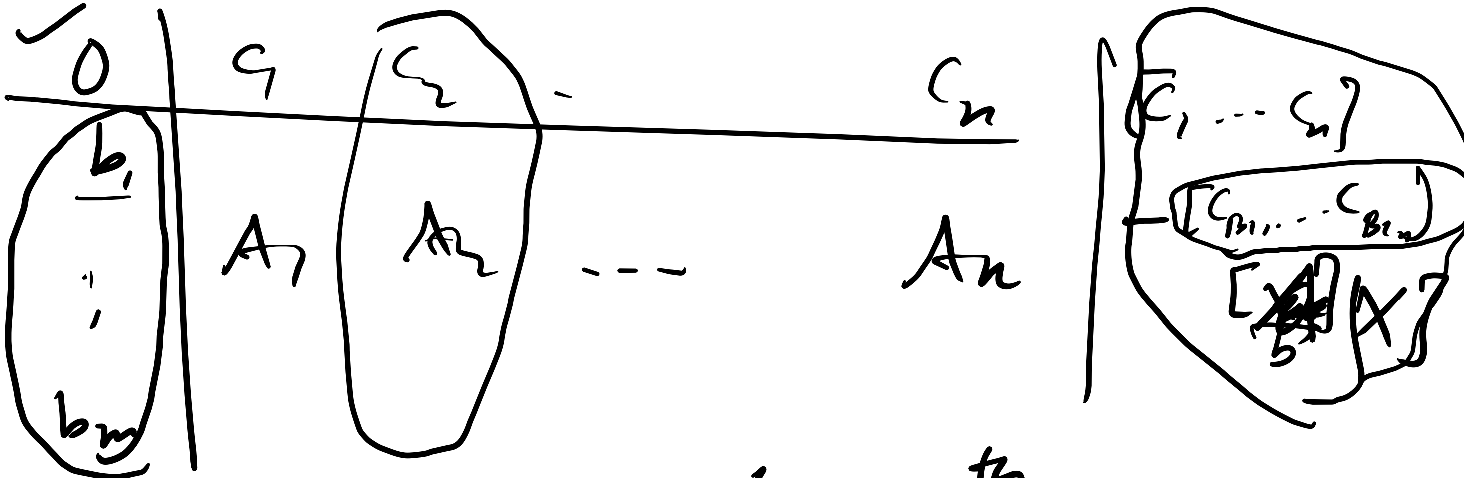
$$\langle \vec{c}, \vec{y} \rangle \geq \langle \vec{c}, \vec{x_0} \rangle \quad \forall \ \vec{y} \in F.$$

$\blacksquare$

* The operation of pivoting, in a geometric sense, takes a vertex to a <u>neighboring</u> vertex.

<u>Corollary</u>: For all $1 \leq i \leq m$, $\bar{c}_{B(i)} = 0$.

that is, the modified cost of a basic column is 0.



Put cost function as the 0th row.

(1) Write c_j in $(0,j)^{th}$ position, and place 0 in cell $(0,0)$.

(2) Let $B(1), \ldots, B(m)$ correspond to a bfs.

Place e_i in column $B(i)$ via elementary row operations.

(3) If j is basic, $\bar{c}_j = 0$. $\boxed{\text{row}(0) \to \text{row}(0) - c_{B(i)}\,\text{row}(i) \text{ for } i = 1, \ldots, n}$

(4) For $j \geq 1$, j^{th} entry of row 0 is

$$c_j - \sum_{i=1}^{m} x_{ij}\, c_{B(i)} = \bar{c}_j$$

(j^{th} modified cost)

~~column 0 is~~ $(0,0)^{th}$ entry here

$$-\left(\sum_{i=1}^{m} x_{i0}\, c_{B(i)}\right) = -(\text{cost of current bfs})$$

$$[0 \quad c_1 \quad \dots \quad c_n] - [c_{B(1)} \quad \dots \quad c_{B(m)}] \underbrace{[\vec{b} \mid X]}_{m \times (n+1) \text{ matrix}}$$

− cost of current bfs	$\bar{c}_1 \quad \bar{c}_2 \quad \dots \quad \bar{c}_n$
$\vec{b}$	X

Example: Min $10x_1 + 2x_2 - 4x_3 + x_4 - 2x_5$

$$
\begin{array}{c|c|ccccc}
 & & & & c & & \\
(\mathrm{I}) & 0 & 10 & 2 & -4 & 1 & -2 \\
\hline
 & 8 & 8 & 4 & -12 & 0 & 0 \\
 & 45 & 5 & 2 & 4 & 11 & 10 \\
 & 21 & 1 & 1 & -2 & 4 & 5 \\
 & b & & & A & &
\end{array}
$$

(2) $B(1) = 3$, $B(2) = 4$, $B(3) = 5$

$B(1) \to e_1$, $B(2) \to e_2$, $B(3) \to e_3$

-1

Corresponds bfr

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{array}{c|ccccc}
0 & 10 & 2 & -4 & 1 & -2 \\
\hline
2 & 2 & 1 & -3 & 0 & 0 \\
1 & 1 & 0 & 8/3 & 1 & 0 \\
3 & -1 & 0 & -29/15 & 0 & 1
\end{array}$$

(3)	1	3	0	$\boxed{-68/15}$	0	0
	2	2	1	-3	0	0
	1	1	0	8/3	1	0
	3	-1	0	-29/15	0	1

$-4 + \frac{8}{15} = \frac{-68}{15}$

$\bar{c}_j = c_j - \sum_{i=1} c_{B(i)} x_{ij}$

$2 \times -3 + \frac{8}{3} + 58/15 = \frac{-90 + 40 + 58}{15} = +8/15$

$$\sum_{i=1}^{m} c_{B(i)}\, x_{ij} = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= 4 + 1 + 2 = 7$$

$$\bar{C}_1 = 10 - 7 = 3$$

Cost of current basic feasible solution

$= -1$

Lemma: The $(0,0)^{th}$ entry will always hold the negative of the cost of the current bfs and the $(0,j)^{th}$ will always hold $\bar{c}_j$ (for $j>0$), (if whenever we pivot we perform a row operation so as to zero out the entry atop the entering column.)

How to pivot:

(1) Find a column j whose row 0 entry is strictly negative.

(if there is no such column, we are done)

(2) Compute $\theta_0 = \min\limits_{\underline{\underline{\{i : x_{ij} > 0\}}}} \dfrac{x_{i0}}{x_{ij}}$ (if $x_{ij} \leq 0$ $\forall i$, then LPS is unbounded!)

↳ assume nonempty

(3) Find an l such that

$$\theta_0 = \frac{x_{l0}}{x_{l1}} \quad (\text{and } x_{lj} > 0)$$

(discard column $B(l)$ from bfs and include column j in bfs)

(Modulo the theorem mentioned in step 2, we have a recipe for pivoting: