21/10/2021

Lecture-9

$P = \{ \vec{x} \in \mathbb{R}^{n} \mid \underline{\underline{A}}\vec{x} \leq \vec{b} \}$

$\vec{v} \in P$ is an extreme point of P

if and only if there are n linearly

independent constraints amongst $\boxed{A}\vec{x} \leq \vec{b}$

that satisfy equality at $\vec{v}$.

$[a_{11} \ldots a_{1n}] \underline{\underline{\vec{v}}} = b_1$

<u>LPS</u>: $\boxed{\langle \vec{c}, \vec{x} \rangle}$ minimize this objective function on F (polyhedron in $\mathbb{R}^n$)

Minimize $\langle \vec{c}, \vec{x} \rangle$ over F using a computational algorithm.

(1) What data structure can store a complete description of F: $(A\vec{x} \leq \vec{b})$

<u>Ans</u>. Specify two arrays, $A \to$ array of size $m \times n$

$\vec{b} \to$ array of size $m \times 1$

(2) How does a computer decide whether F is empty or not?

(3) If F is nonempty, how does a computer decide whether the LP is bounded or not? ✓ question which we will answer in today's lecture.

(4) If F is nonempty and the LP is bounded, how does a computer find an optimal solution? (Simplex Method)

Theorem: In LPS, suppose that $\vec{p} \in F$. Then either the LPS is unbounded or there is a vertex $\vec{v}$ of F satisfying

$$\langle \vec{c}, \vec{v} \rangle = \vec{c}^{\,T} \vec{v} \leq \langle \vec{c}, \vec{p} \rangle = \vec{c}^{\,T} \vec{p}.$$

Proof: $F = \{ \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{b} \text{ and } \vec{x} \geq \vec{0} \}$

$$A\vec{p} = \vec{b} \quad \text{and} \quad p_i \geq 0 \quad \forall\, i \in \{1, \ldots, n\}$$

(1) $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \vec{x} \geq \vec{0}$ → $n \times n$ identity matrix

$A\vec{x} = \vec{b}$ → $m \times n$ matrix, $m \leq n$

$$\begin{array}{c} I \\ I \end{array} \begin{bmatrix} I_n \\ A \end{bmatrix} \to \underline{\underline{(m+n)}} \times n$$

Take all sets of linearly independent constraints that contain the constraints $A\vec{x} = \vec{b}$ (and all of which are __tight__ at $\vec{p}$.)

Choose a largest one $S \subseteq \{1, 2, \ldots, m+n\}$.

$\{n+1, \ldots, m+n\}$

Let $r = |S|$.

$I = S \cap \{1, 2, \ldots, n\}$

consists of the $r-m$ sign constraints in S.

(2) If $r = n$, $\vec{p}$ is a vertex of F.

(3) If $m \leq r < n$, we will show that either

$\langle \vec{c}, \vec{x} \rangle$ (for $\vec{x} \in F$) can be made arbitrarily small, OR

[$\exists\ \vec{p^*} \in F$ such that $\langle \vec{c}, \vec{p^*} \rangle \leq \langle \vec{c}, \vec{p} \rangle$ for which there is a larger set of linearly independent constraints tight at $\vec{p^*}$.]

* Iterating the process at most $n-m$ times, we either find a proof that the cost is

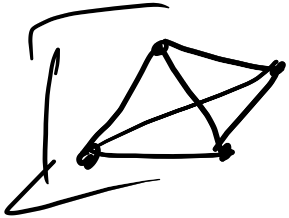
is <u>unbounded</u>, or we reach a vertex $\vec{v}$ such that $\langle \vec{c}, \vec{v} \rangle \leq \langle \vec{c}, \vec{p} \rangle$.

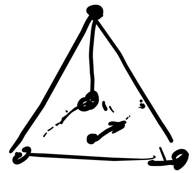
<u>Observation</u>: Since S was the largest set of indices,

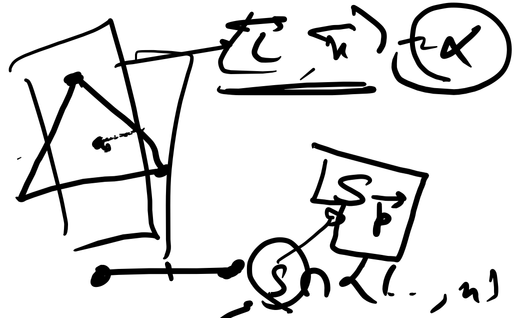
if $b_j = 0$

$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ m+n \end{array} \begin{bmatrix} I_n \\ \hline A \end{bmatrix} \geq \begin{bmatrix} \vec{0} \\ \hline \vec{b} \end{bmatrix}$$

for $j \notin S$ if $x_j \geq 0$ is a sign constraint tight at $\vec{p}$, then it must be linearly dependent on the tight constraints corresponding to S.







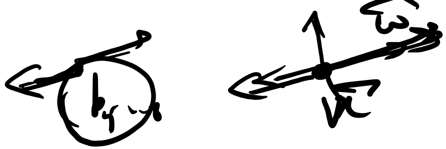
Let

$F^{\alpha} = \{ \vec{x} \in \mathbb{R}^n \mid \boxed{x_j = 0} \ \forall j \in I$ and $A\vec{x} = \vec{0} \}$

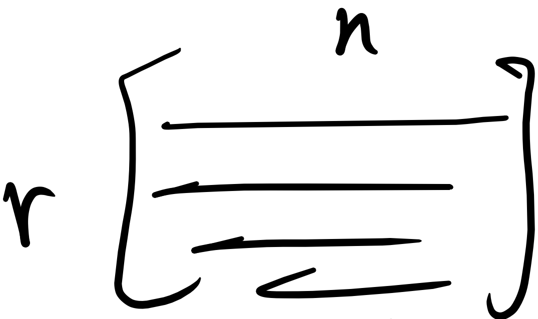
$(\vec{p} \in F^{\alpha}) + \vec{I}\}$

$A(\vec{x} + \lambda\vec{w}) = A\vec{x} = \vec{I}$

$\vec{w} \in F^{\alpha}, \ \vec{x} \in T$



$\boxed{\dim(F^*) = n - r}$ (rank-nullity theorem)



$\vec{x} = \vec{0}$

rank $= r$.

[If $r < n$, there is a non-zero vector $\vec{w} \in F^*$.]

Assume $\boxed{\langle \vec{c}, \vec{w} \rangle \leq 0}$. (by negating $\vec{w}$ if necessary)

Claim: If $p_j = 0$, then $w_j \geq 0$.

Proof of Claim: ~~If $p_j = 0$ and $j \notin I$~~

($w_j = 0$ for $j \in I$ (by def^n of F^*))

If $p_j = 0$ and $j \notin I$, then the constraint $x_j \geq 0$, tight at $\vec{p}$, must be linearly dependent on the constraints on S.

$$\begin{array}{c} j \\ \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} w \end{bmatrix} = w_j$$

$$= \sum_{i \in S} \lambda_i \vec{a}_i \quad \text{rows of } \begin{bmatrix} I_n \\ \hline A \end{bmatrix}$$

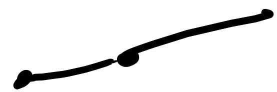
$$\langle \vec{a}_i, \vec{w} \rangle = 0 \quad \text{for } i \in S$$

$$w_j = \langle e_j, \vec{w} \rangle = \Big\langle \sum_{i \in S} \lambda_i \vec{a}_i, \vec{w} \Big\rangle = \sum_{i \in I} \lambda_i \underbrace{\langle \vec{a}_i, \vec{w} \rangle}_{=0} = 0$$

For all $\lambda \in \mathbb{R}$, $\boxed{A(\vec{p} + \lambda \vec{w}) = \vec{b}}$.

(as $A\vec{w} = \vec{0}$)

If $p_j = 0$, then $(\vec{p} + \lambda \vec{w})_j = 0$

* For $\lambda \in \mathbb{R}$, $\vec{p} + \lambda \vec{w}$ 

satisfies the equality constraints of LPs

To Prove: One of the following two possibilities happens:

(I) We may choose λ arbitrarily large in such a way that $\vec{p} + \lambda \vec{w} \in F$ and $\langle \vec{c}, \vec{p} + \lambda \vec{w} \rangle$ can be made arbitrarily small. (~~though~~ $\langle \vec{p}, \vec{w} \rangle \geq 0$)

(II) $\exists \, \lambda \in \mathbb{R}$ such that $\langle \vec{c}, \vec{p} + \lambda \vec{w} \rangle \leq \langle \vec{c}, \vec{p} \rangle$ and at least one more sign constraint is tight at $(\vec{p} + \lambda \vec{w})$ than at $\vec{p}$ (and that sign constraint is linearly independent from the constraints on S.)

Let $J = \{ j : \underline{w_j < 0} \} \subseteq \{1, \dots, m+n\}$ (?)

Case I : Again $\boxed{\langle \vec{c}, \vec{w} \rangle < 0}$ and $J = \phi$.

that is, $\boxed{w_j \geq 0}$ $\forall j \in \{1, \dots, m+n\}$.

So, $\vec{p} + \lambda \vec{w} \in F$ for all $\boxed{\lambda \geq 0}$,

$$\langle \vec{c}, \underbrace{\vec{p} + \lambda \vec{w}}_{\in F} \rangle = \underline{\langle \vec{c}, \vec{p} \rangle} + \lambda \langle \vec{c}, \vec{w} \rangle$$

< 0

<u>Case II</u>: $\underline{w_j < 0}$ for some j. $(J \neq \phi)$.

Let $s = \max\limits_{j \in J} \dfrac{p_j}{-w_j}$

Note: $p_j > 0$ for $j \in J$.

Thus $\boxed{s > 0}$. Choose j^* such that

$s = \dfrac{p_{j^*}}{-w_{j^*}}$ and define $\left[\vec{p^*} = \vec{p} + s\vec{w} \right]$

For all $j \in J$,

$$p_j^* = p_j + s w_j \geq p_{j'} + \frac{p_j}{-w_j} \cdot w_j$$

$$= 0$$

For $j \notin J$, $p_j^* \geq 0$, and

$$p_{j^*}^* = p_{j^*} + \frac{p_{j^*}}{-w_{j^*}} \cdot w_{j^*} = 0$$

At j^*, the sign constraint has become tight.

(Both $\vec{p}$ and $\vec{p}^*$ satisfy all the constraints in S with equality. If the constraint $x_{y^*} \geq 0$ were dependent on S,

$$p^*_{y^*} = \nabla p_{y^*}.$$

$A_I \vec{x} = \vec{0}$

$A \vec{x} = \vec{b}$)

$S \to S \cup \{y^*\} \to \nexists$ new r^a ~~new~~ new $\vec{w}$.

$\vec{p} \to \vec{p}^*$

Case III: $\langle \vec{c}, \vec{w} \rangle = 0$, $J = \emptyset$.

$$\left(\text{for all } j,\ \boxed{w_j \geq 0} \right)$$

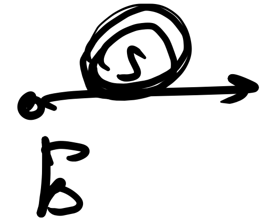
Let
$$s = \min_{\boxed{w_j > 0}} \frac{p_j}{w_j}$$

nonempty as $\vec{w}$ is a non-zero vector.

$$s = \frac{p_{j^*}}{w_{j^*}} \quad \text{for some } j^* \in S.$$

$$(p_{j^*} > 0,\ w_{j^*} > 0) \qquad \boxed{s > 0}$$

$$\vec{p}^* = \vec{p} - s\vec{w}$$



$$A\vec{p}^* = A\vec{p} - sA\vec{w}$$

$$= A\vec{p} = \vec{b}$$

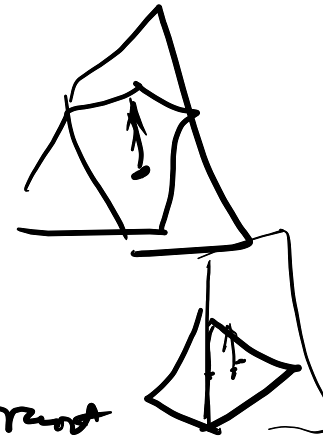
$$p_j^* = p_j - s w_j \geq p_j - \frac{p_j}{w_j} w_j = 0$$

$$p_{j^*}^* = p_{j^*} - \frac{p_{j^*}}{w_{j^*}} \cdot w_{j^*} = 0$$

$\vec{p^*}$ is feasible.

$\boxed{p^*_{y^*} = 0}$ while $p_{y^*} > 0$.

$x_{y^*} \geq 0$ tight at $\vec{p^*}$.



So, $\vec{p^*}$ satisfies one more sign constraint which is linearly independent from the constraints on S. $\langle \vec{c}, \vec{p^*} \rangle = \langle \vec{c}, \vec{p} \rangle - s \underline{\underline{\langle \vec{c}, \vec{u} \rangle}}$

$= \langle \vec{c}, \vec{p} \rangle$ ∎

__Pseudocode__ : $\vec{p}$

(Check $A\vec{p} = \vec{b}$ and $\vec{p} \geq \vec{0}$)

(1) Using Gaussian elimination... $\left[\begin{array}{c} I_n \\ \hline A \end{array}\right]$

find $S_{\vec{p}}$ such that

$\mathcal{I}_{\vec{p}} \cup \{n+1, \dots, m+n\}$

$A\vec{p} = \vec{b}$ and $p_j = 0$ for $j \in \mathcal{I}_{\vec{p}}$, $S_{\vec{p}} \cap \{1, \dots, n\}$

If number of linearly independent constraints that are tight at $\vec{p}$ (?) and contain $A\vec{x} = \vec{b}$ is n ($\#(S_{\vec{p}}) = n$)

__terminate__ $\boxed{\#(\mathcal{I}_{\vec{p}}) = n - m}$

and $\boxed{\text{return } \vec{b} \text{ and } \langle \vec{c}, \mathcal{I} \rangle}$

$\vec{b}$: optimal soln

$\langle \vec{c}, \mathcal{I} \rangle$: optimal cost.

(2) Find non-zero vector

$$\vec{w} \in \langle \vec{x} \in \mathbb{R}^n \mid x_j = 0 \ \forall j \in \mathcal{I}, \ \boxed{A\vec{x} = \vec{0}} \rangle$$

using Gaussian elimination such that

$$\langle \vec{c}, \vec{w} \rangle \leq 0.$$

(3) Find $J := \{ j : w_j > 0 \}$.

(4) If $\langle \vec{c}, \vec{w} \rangle < 0$ and $J = \emptyset$:
conclude that LPS instance
is unbounded, and terminate. Return
"LPS is unbounded."

(5) else if $J \neq \emptyset$:
$s := \max_{j \in J} \frac{p_j}{-w_j}$, $j^* := \arg\max_{j' \in J} \frac{p_j}{-w_j}$

replace $\vec{p}$ with $\vec{p} + s\vec{w}$, and $I_{\vec{p}}$ with $I_{\vec{p}} \cup \{j^*\}$.

⑥ else if $(\langle \vec{c}, \vec{\omega} \rangle \geq 0,\ J = \phi)$

replace $\vec{b}$ with $\vec{b} - \delta\vec{\omega}$

$\vec{J}_{\vec{b}}$ with $\vec{J}_{\vec{b}} \cup \{j^*\}$ where

$$\delta = \min_{\{j : \omega_j > 0\}} \frac{b_j}{\omega_j}, \quad j^* = \arg\min_{\{j : \omega_j > 0\}} \frac{b_j}{\omega_j}$$

Go to step (1).