30/09/2021

Lecture - 4

Decision space: $\boxed{A\vec{x} \leq \vec{b}}$ → convex set

→ convex cone

$C\vec{x} = \vec{d}$

→ affine set

halfspaces: one nontrivial linear inequality

hyperplane: one " linear equality

polyhedron: finitely many linear equalities and linear inequalities

polytope: bounded polytopes

Caratheodory's theorem

$S \subseteq \mathbb{R}^n$, $\vec{x} \in \mathrm{conv}(S)$.

$\vec{x}$ is a convex combination of affinely independent points of S.

$$\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{x}_i$$

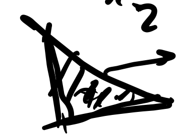
k-simplex: an example of polytope.

$$x_1 + \dots + x_k = 1$$
$$x_1 \geq 0$$
$$\vdots$$
$$x_k \geq 0.$$

$x_1 + x_2 = 1$

$x_1 \geq 0$

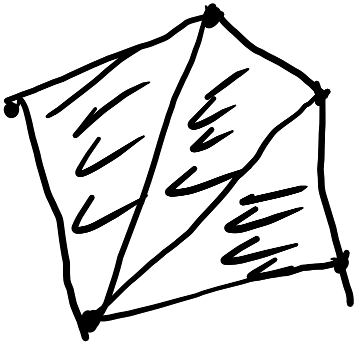
$x_2 \geq 0$



$S = \bigcup$ simplices in S

given $\vec{x}$ in conv(S).

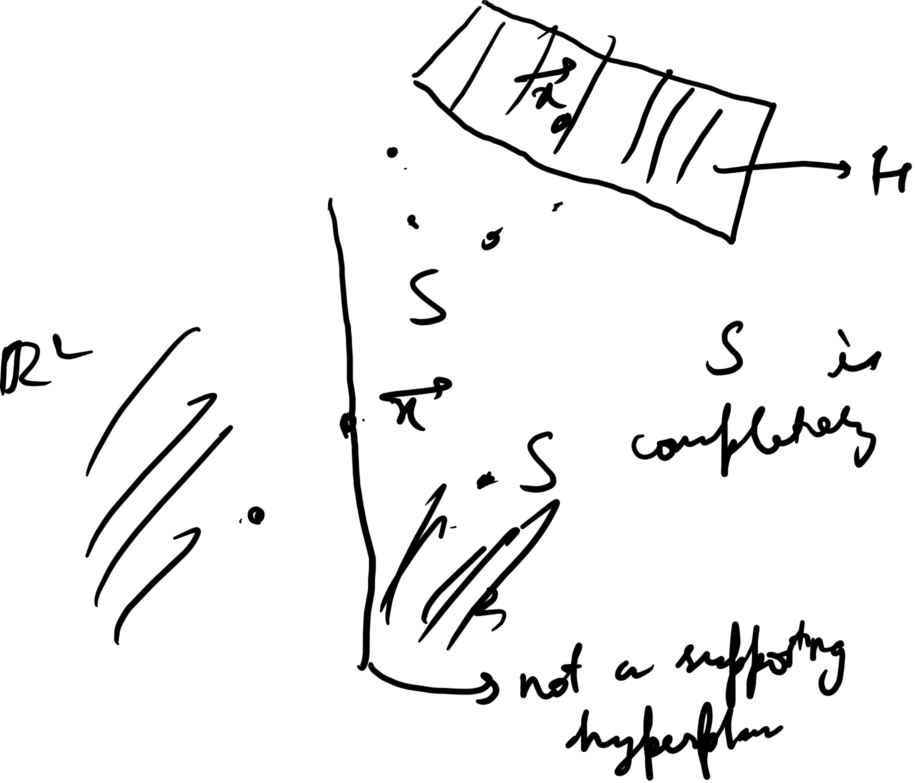
$\Rightarrow \vec{x} \in$ some simplex with vertices in S.



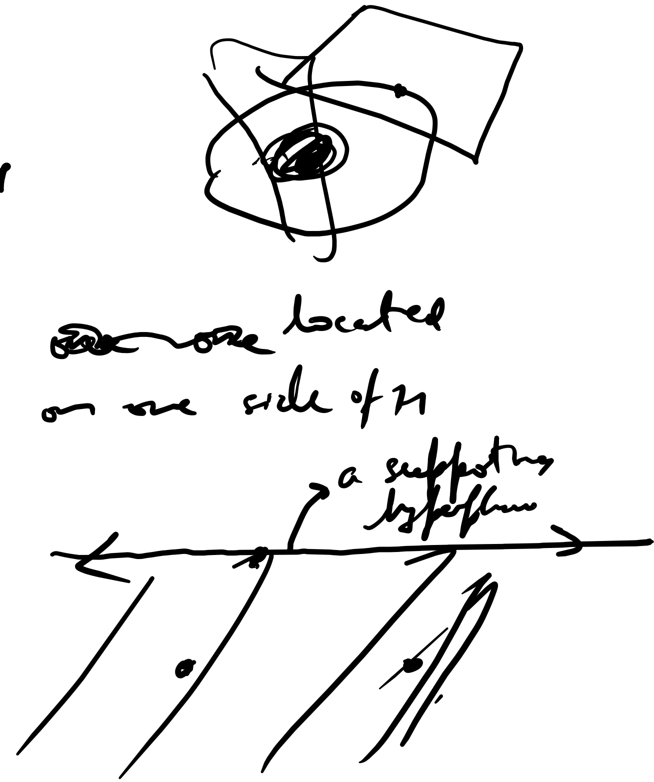
Representation theorem (; ~~For~~ Each nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspace.

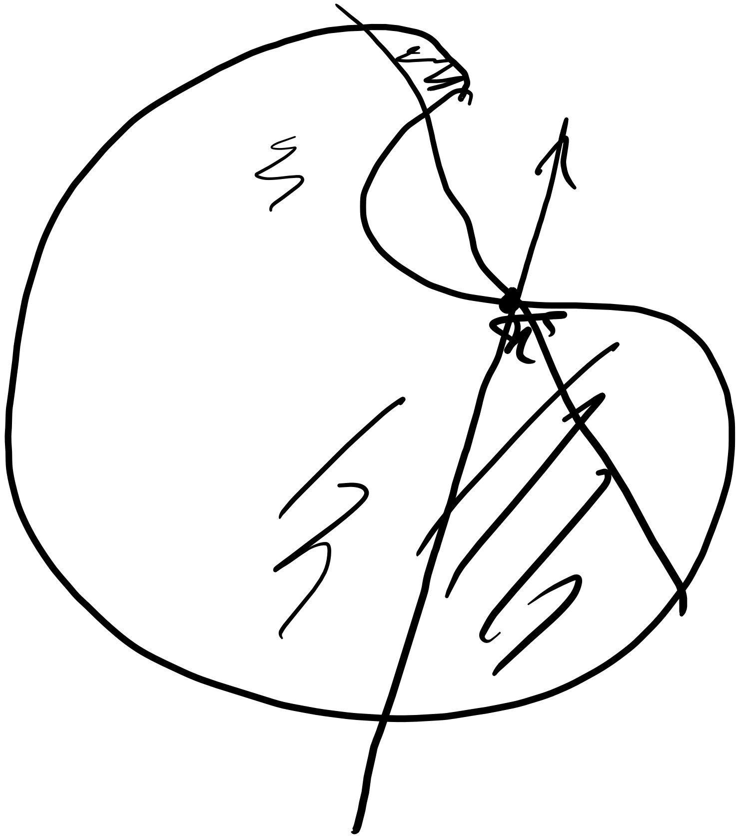
Defn. Let $S \subseteq \mathbb{R}^n$ be a **subset** of $\mathbb{R}^n$, and $H = H_{\vec{a},b} \subseteq \mathbb{R}^n$ be a hyperplane, and let $H^+_{\vec{a},b}$ and $H^-_{\vec{a},b}$ denote the two close halfspaces bounded by $H_{\vec{a},b}$.

We say that H **supports** S at $\vec{x} \in \mathbb{R}^n$ if $\vec{x} \in S \cap H$ and either $S \subseteq H^+_{\vec{a},b}$ or $S \subseteq H^-_{\vec{a},b}$

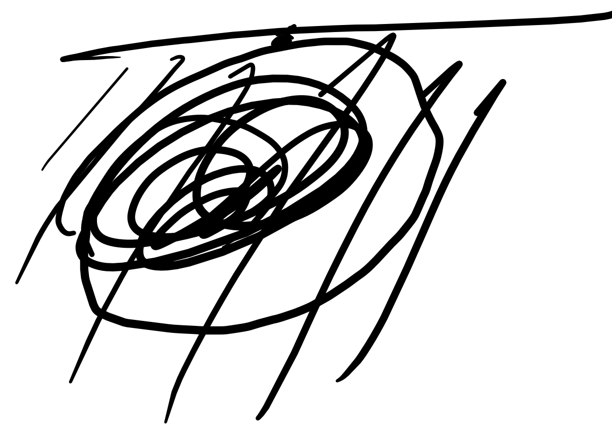


S is ~~one~~ located completely on one side of H



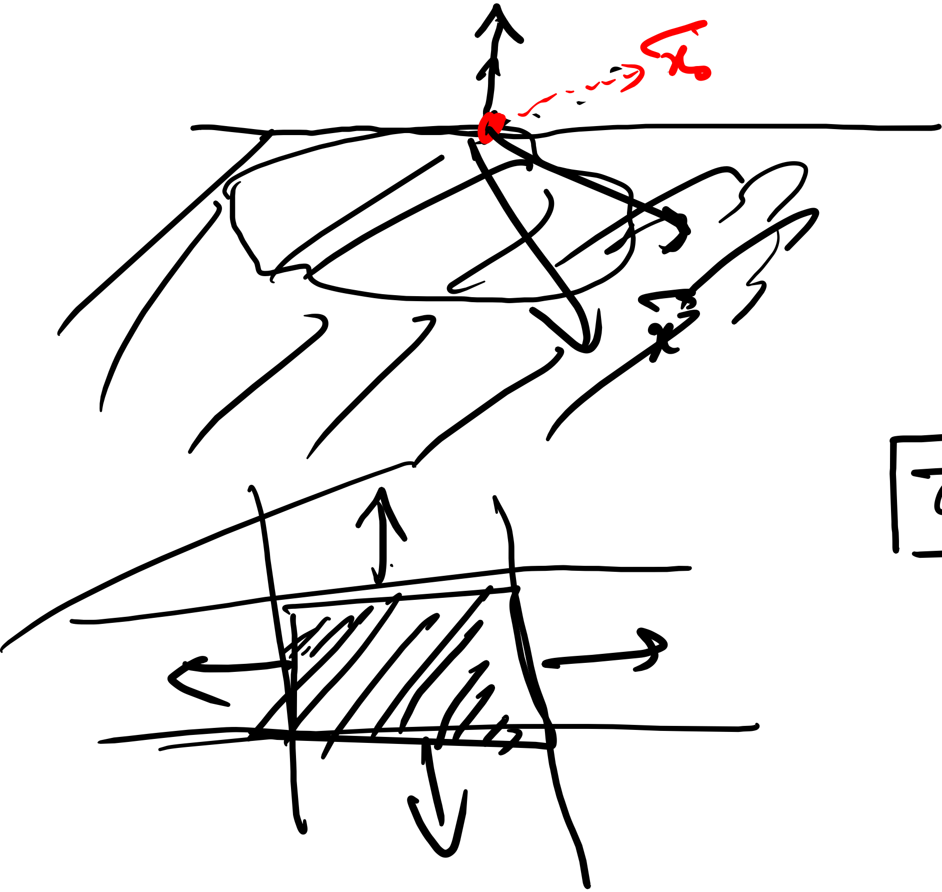


$\mathbb{R}^n$



H is a <u>support plane</u> of S or <u>supports</u> S, if H supports S at some point $\bar{x} \in \mathbb{R}^n$, which is necessarily a boundary point.

If $H_{\vec{a},b}$ supports S and $S \subseteq H^{-}_{\vec{a},b}$ ($H^{+}_{\vec{a},b}$, respectively) then $H^{-}_{\vec{a},b}$ ($H^{+}_{\vec{a},b}$, respectively) is called a supporting half-space of S at $\bar{x}$. (and $\vec{a}$ is called an exterior or outer normal vector), in case $S \subseteq H^{-}_{\vec{a},b}$)



$\vec{a}^T \vec{x} \le b$

$\langle \vec{a}, \vec{x} \rangle \le b$

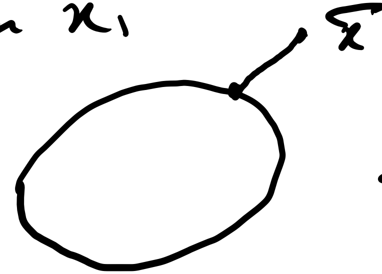
$\langle \vec{a}, \vec{x} - \vec{x}_0 \rangle \le 0$

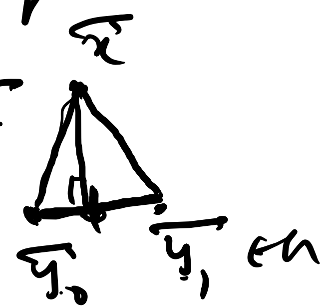
$\boxed{\vec{a}}$ points outwards to the halfsp

Metric projection.

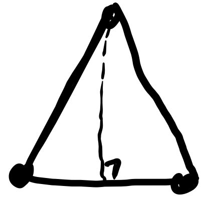
$K \to$ fixed nonempty closed convex set.

Lemma: To each $\vec{x} \in \mathbb{R}^n$, there exists a unique point $p(K, \vec{x}) \in K$ satisfying $|\vec{x} - p(K, \vec{x})| \leq |\vec{x} - \vec{y}|$ $\forall \vec{y} \in K$, that is, there is a unique closest point on K from $\vec{x}$.





Uniqueness.



$$\left|\vec{x} - \frac{\vec{y_0}+\vec{y_1}}{2}\right| \qquad < \vec{y_0}, \quad < \vec{y_1}$$

$$\leq \frac{1}{2}\left|\vec{x}-\vec{y_0}\right| + \frac{1}{2}\left|\vec{x}-\vec{y_1}\right|$$

$$= \frac{1}{2}\left|\vec{x}-\vec{y_0}\right|$$

Existence.

Choose $\rho > 0$ s.t. $B(\vec{x}, \rho) \cap K \neq \emptyset$.

$B(\vec{x}, \rho) \cap K$ is compact.

$d(\vec{x}, B(\vec{x}, \rho) \cap K)$ is continuous

on $B(\vec{x}, \rho) \cap K$ and attains a minimum.

Corollary: To each $\vec{x} \in \mathbb{R}^n$, there is a unique point $p_H \vec{x} \in H$ which is closest to $\vec{x}$ (H is a hyperplane).

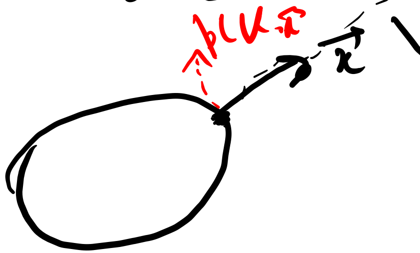
$\cancel{p(K,\vec{x})}$

Defn. The mapping $p(K,\cdot) : \mathbb{R}^n \to K$ is called the metric projection or nearest point map of K.

For $\vec{x} \in \mathbb{R}^n \setminus K$, we denote

$$u(K,\vec{x}) = \frac{\vec{x} - p(K,\vec{x})}{d(K,\vec{x})}$$

unit vector pointing from $p(K,\vec{x})$ to $\vec{x}$.



$$R(K,\vec{x}) = \{\, p(K,\vec{x}) + \lambda u(K,\vec{x}) : \lambda \geq 0 \,\}$$

ray through $\vec{x}$ with endpoint $p(K,\vec{x})$.

<u>Theorem</u>: The metric projection is contracting

that is, $|p(K,\vec{x}) - p(K,\vec{y})| \leq |\vec{x} - \vec{y}|$

$\forall\, \vec{x}, \vec{y} \in \mathbb{R}^n$.

<u>Proof</u>: Assume $p(K,\vec{x}) \neq p(K,\vec{y})$.

$p(K,\vec{x}) - p(K,\vec{y}) \neq \vec{0}$.

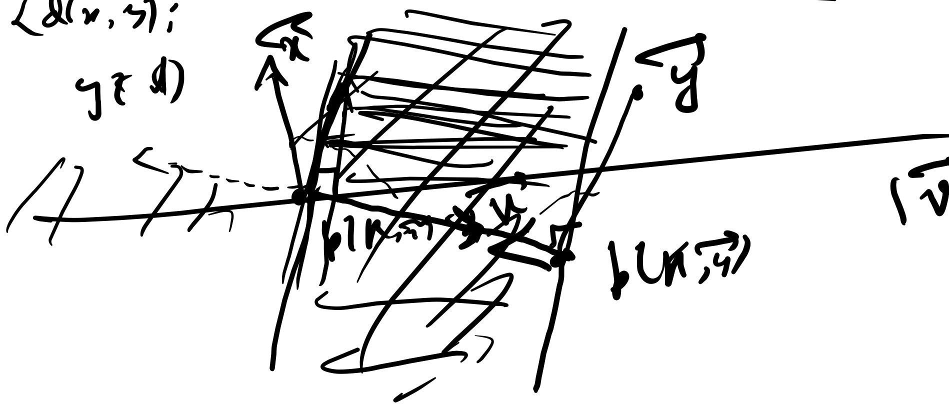
$$\vec{v} := p(K,\vec{x}) - p(K,\vec{y})$$

$$\langle \vec{x} - p(K,\vec{x}), \vec{v} \rangle \geq 0$$

$$\langle \vec{y} - p(K,\vec{y}), \vec{v} \rangle < 0$$

$d(x, A) = \inf \{ d(x,y) ;\ y \in A \}$

$d(x_1, x_2)$



$|\vec{v}| \leq |\vec{y} - \vec{x}|$

The line segment $[\vec{x}, \vec{y}]$ meets the two hyperplanes that are orthogonal to $\vec{v}$ and that go through $p(K, \vec{x})$ and $p(K, \vec{y})$, respectively.

Lemma: Let S be a sphere containing K in its interior. Then, $p(K, S) = \mathrm{bd}\, K$.

Proof:

For $\vec{x} \in \mathrm{bd}\, K$, $\exists\ \vec{y} \in \mathrm{bd}\, S$, s.t.

$$\boxed{p(K, \vec{y}) = \vec{x}}.$$

Proof. $p(K, S) \subseteq \operatorname{bd} K$.

Let $\vec{x} \in \operatorname{bd} K$. For $i \in \mathbb{N}$, choose $\vec{x}_i \in \operatorname{int}(S) \setminus K$ such that

$$|\vec{x}_i - \vec{x}| < \frac{1}{i}$$

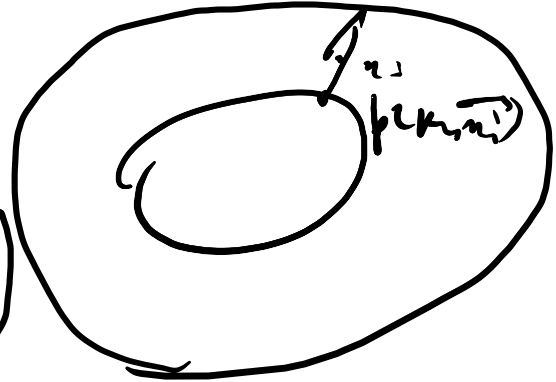
As $p(K, \cdot)$ is contracting,

$$|\vec{x} - p(K, \vec{x}_i)| = |p(K, \vec{x}) - p(K, \vec{x}_i)|$$

$$\leq |\vec{x} - \vec{x}_i| < \frac{1}{i}$$

The ray $R(K, \vec{x}_i)$ meets S at $\vec{y}_i$.

and $\boxed{p(K, \vec{y}_i) = p(K, \vec{x}_i)}$



Hence,

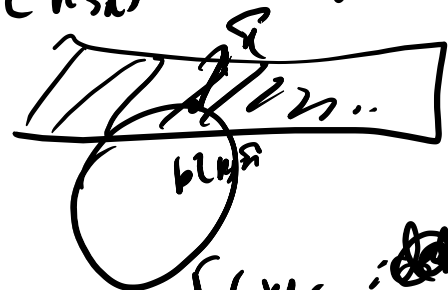
$$|\vec{x} - p(K, \vec{y}_i)| < \frac{1}{i}$$

$\vec{y}_i$ is a sequence of points on the compact set $bd(S)$

$\Rightarrow$ there is a convergent subsequence which converges to $\vec{y} \in bd(S)$.

$$\boxed{\vec{x} = p(K, \vec{y})}$$

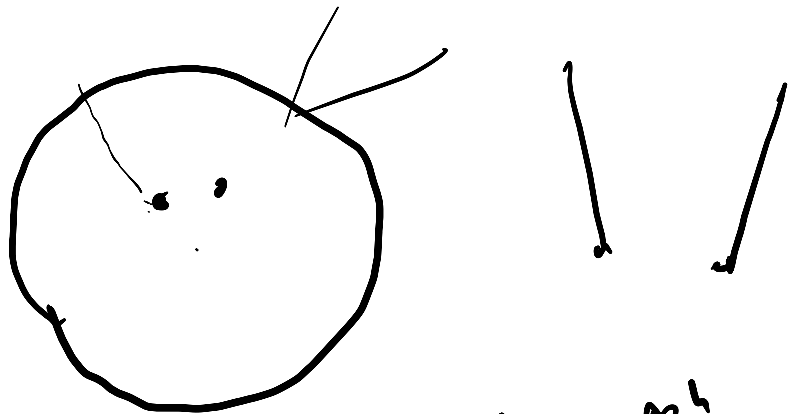
(<u>Lemma</u>. K nonempty closed convex set in $\mathbb{R}^n$. $\vec{x} \in \mathbb{R}^n \setminus K$. The hyperplane H through $p(K, \vec{x})$ orthogonal to $\vec{x} - p(K, \vec{x})$ supports K.)



$f(x) =$ [illegible]

$d(x)$ – closest point in A to x.

$d(f(x), f(y)) \leq d(x, y).$



$d(f(x), f(y)) \geq d(x, y)$

$\leq 2\, d(x, y)$

$p(k, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$

k^4

$d_K : \mathbb{R}^n$

$\bar{x} \mapsto \underline{\underline{d(\bar{x}, p(k, \bar{x}))}}$