24/02/2022

Lecture - 10

$\boxed{\text{separating family of seminorms on } V}$ $\longleftrightarrow$ locally convex topologies on V

Theorem. $V \to \mathbb{K}$-space.

$\Gamma \to$ separating family of seminorms on V.

(1) Then there is a locally convex topology on V in which, for each $x_0 \in V$, the family of sets

$$\left[V(x_0; p_1, \ldots, p_m, \varepsilon) = \{ x \in V : p_j(x - x_0) < \varepsilon, \ j = 1, \ldots, m \} \right]$$

(where $\varepsilon > 0$, $p_1, \ldots, p_m \in \Gamma$), is a base of nbhds of x_0.

(ii) With this topology, each of the seminorms on Γ is continuous. (This is the coarsest topology such that all the seminorms on Γ are continuous).

(iii) Every locally convex topology arises in this way from a suitable separating family of seminorms.

Proof - Step I : $\{ V(x_0; p_1, \ldots, p_m; \varepsilon) ; x_0 \in V, p_1, \ldots, p_m \in \Gamma, \varepsilon > 0 \}$ is a base for a topology τ on V.

Step II : τ is a Hausdorff topology.

Step III : (V, τ) is a topological vector space.

$(x, y) \longmapsto x + y$, are continuous
$V \times V \longrightarrow V$ maps w.r.t. τ.
$(a, x) \longmapsto a \cdot x$
$\mathbb{K} \times V \longrightarrow V$

Step IV : τ is locally convex.

Each of the basic nbds. $V(x_0; p_1, \dots, p_m; \varepsilon)$ is convex.

$x, y \in$ $\quad p_j(x - x_0) < \varepsilon \quad j = 1 \dots m$

$p_j(y - x_0) < \varepsilon$

$$p_j(\lambda x + (1-\lambda) y - x_0) \leq \lambda p_j(x - x_0) + (1-\lambda) p_j(y - x_0)$$
$$< \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon$$

Step V ; $p \in P$ is uniformly continuous on V. $\boxed{V(\tau_0; p, 1)}$

Claim : Suppose that V is a TVS, and p is a semi-norm on V. If p is bounded on some nbd. of 0 in V, then p is (uniformly) continuous on V.

Proof of Claim : $p(U) \subseteq [-b, b]$ for some open set U containing 0. $b \in \mathbb{R}_{>0}$.

$\varepsilon > 0$, $\varepsilon b^{-1} U$ is a nbd. of 0 s.t.

$$|p(\varepsilon b^{-1} U)| \leq \boxed{\varepsilon}.$$

$$|p(y) - p(x)| \leq p(y-x) < \varepsilon$$

for $x, y \in V$ and $y - x \in \varepsilon b^{-1} V$.

(iii) Start with a locally convex topology τ.

$\Gamma_0 \to$ set of all τ-continuous seminorms on V,

Step I: Every τ-nbd U of 0 contains a set of the form $\{x \in V : p(x) < 1\}$ for some $p \in \Gamma_0$.

U be a τ-nbd of 0

U contains a convex τ-nbd. $\boxed{U_0}$ of 0

U_0 contains a balanced τ-nbd. $\boxed{U_1}$ of 0.

conv(U_1) $\longrightarrow$ convex hull of U_1.

~~If~~ $\Downarrow$ is the union of the sets of the form

$\boxed{a_1 x_1 + \dots + a_{n-1} x_{n-1} + a_n U_1}$, ($a_1 + \dots + a_n = 1$

$0 \le a_i \le 1$)

$a_n \neq 0$)

conv(U_1) is open, convex and

contained in $U_0 \subseteq U_1$

Since U_1 is balanced, conv(U_1) is also balanced.

conv(U_1) — open, convex, balanced nbd of 0 contained

in U.

There is a seminorm p on V s.t.

$$\boxed{\operatorname{conv}(U)} \subseteq \{ x \in V ; p(x) < 1 \} \subseteq U .$$

p is τ-continuous.

Step II. Γ_0 is separating.

If $x \in V$ and $p(x) = 0$ for every p in Γ_0, then x lies in every τ-nbd of 0, since τ is Hausdorff; $x = 0$.

Step III. $V(x_0, p_1, \dots, p_m, \varepsilon)$ is a τ-nbd of x_0, $p_1, \dots, p_m \in \Gamma_0$, $\varepsilon > 0$.

(LC topology obtained from Γ_0 is coarser than τ)

($\boxed{\tau}$ is coarser than the LC topology obtained from Γ_0.)

$$\boxed{V(x_0, p, 1)} = \{x_0 + x \,;\, x \in V,\ p(x) < 1\} \overset{??}{\subseteq} x_0 + U,$$

Since $p_1, \ldots, p_m$ are τ-continuous,

$p : V \to \mathbb{R}$ $[0,1)$

$V(x_0, p, 1)$ $\Leftarrow$ $p(U) \subseteq [0,1)$ $x \in U$ $\Rightarrow$ $p(x) < 1$.

$\Rightarrow p^{-1}([0,1))$ is τ-open, τ-cont.

Corollary: In a LCS, there is a base of nbds of 0 consisting of balanced convex sets.

Proof: $V(0; p_1, \ldots, p_n; \varepsilon)$ is balanced and convex.

($p(x) < \varepsilon$, $p(ax) = |a| p(x) < \varepsilon$ if $|a| = 1$)

Continuity and boundedness:

Proposition: Suppose that V_1, V_2 are LCS over the same scalar field. Let Γ_1, Γ_2, resp., be separating families of seminorms on V_1, V_2 respectively that give rise to the topology of V_1, V_2, respectively.

(1) A seminorm p on V_1 is continuous iff and only if there is a positive real number C and a finite set $p_1, \ldots, p_m$ of elements of Γ_1, such that

$$p(x) \leq C \underline{\underline{\max}} \{ p_1(x), \ldots, p_m(x) \}$$

$x \in V_1$.

(2) A linear operator $T: V_1 \to V_2$ is continuous if and only if given any q on Γ_2, there is a positive real number C and a finite set

$p_1, \dots, p_m \in \Gamma$ such that

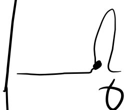
$$q(Tx) \leq C_q \max\{p_1(x), \dots, p_m(x)\}, \quad x \in V_1$$

(3) A linear functional ρ on V_1 is continuous if and only if there is a positive real number C and a finite set $p_1, \dots, p_m$ in Γ_1 such that

$$|\rho(x)| \leq C \max\{p_1(x), \dots, p_m(x)\}$$

$$x \in V_1,$$

$u : S(\mathbb{R}^n) \to \mathbb{R}$



rapidly decreasing smooth functions

$$\left\| x^\alpha \frac{\partial^\beta}{\partial x^\beta} f_{c\ell} \right\|_\infty$$

$\alpha, \beta \in \mathbb{N} \cup \{0\}$

$c\ell_n \to c\ell$

$\forall \alpha, \beta \in \mathbb{N} \cup \{0\}$

$$\overline{x^\alpha \frac{\partial^\beta}{\partial x^\beta} c\ell_n} \to \overline{x^\alpha \frac{\partial^\beta}{\partial x^\beta} c\ell} \text{ in } L^\infty$$

Example. Let $(V, \|\cdot\|)$ be a normed linear space.

$\|x\| < \epsilon$.

$\|x\| = 0 \iff x = 0$.

$\Gamma = \{\|\cdot\|\}$. Then the metric topology w.r.t. $\|\cdot\|$ on V is precisely the locally convex topology w.r.t. Γ.

(ii) $T : (V_1, \|\cdot\|_1) \to (V_2, \|\cdot\|_2)$

T is continuous if and only if there exists a positive real number C s.t.

$$\boxed{\|Tx\|_2 \leq C\|x\|_2} \quad \forall x \in V_1.$$

$$\|T\|_{op} = \sup_{x \neq 0} \left(\frac{\|Tx\|}{\|x\|} \right)$$

$\Gamma = \{p, \ldots\}$

$V \to \Gamma$ is a sep. family of seminorms on V.

$\boxed{x_i \to x}$ What does this mean?

iff and only $\boxed{p(x_i) \to p(x)}$ $\forall p \in \Gamma$.

in $\mathbb{R}$.

Proof: (1) Let p be continuous

$\{x \in V \mid p(x) < 1\}$ is a nbd. of 0 in V.

and thus contains the basic nbd.

$V(0; p_1, \ldots, p_m; \varepsilon)$ $\quad \varepsilon > 0$.

Whenever $\max\{p_1(x), \ldots, p_m(x)\} < \varepsilon$

we have $p(\varepsilon x) < \varepsilon$.

$$\Rightarrow \boxed{p(\varepsilon x) \leq \max\{p_1(x), \ldots, p_m(x)\}}.$$

$$\Rightarrow p(x) \leq \underbrace{\frac{1}{\varepsilon}}_{c} \max\{p_1(x), \ldots, p_m(x)\}.$$

Conversely, if $p(x) \leq c \max\{p_1, \ldots, p_m(x)\}$

p is bounded on $\boxed{V(0; p_1, \ldots, p_m; 1)}$.

$\Rightarrow$ p is uniformly continuous on V_1.

(ii) $\boxed{T : V_1 \to V_2}$ is continuous

if and only if whenever $x_i \to x$, $Tx_i \to Tx$.

To Prove: T is continuous if and only if $q \circ T : V_1 \to \mathbb{R}$ is continuous for every seminorm q in Γ_2.

Assume $q \circ T$ is continuous for every seminorm q on Γ_2.

$\boxed{x_i \to x \text{ in } (V_1, \tau_1)}$ $q \circ \underbrace{Tx_i}_{y_i} \to q \circ \underbrace{Tx}_{y}$ $\forall q \in \Gamma_2$.

$\Leftrightarrow$ $Tx_i \to Tx$ in (V_2, τ_2)

(iii) $\rho : V \to \mathbb{K}$ modulus function)

iff $\exists\, C \geq 0$ and $p_1, \dots, p_m \in P$ s.t.

$$|\rho(x)| \leq C_\rho \max\{p_1(x), \dots, p_m(x)\}.$$

$(V, \|\cdot\|)$

$\rho : V \to \mathbb{C}$ is continuous iff and only if

$$|\rho(x)| \leq C_\rho \|x\| \quad \forall x \in V.$$

$\{\rho_1, \dots, \rho_2, \dots\}$ countable family of seminorms.

$$\cancel{p(x) = p_1(x) + \frac{p_2(x)}{\alpha_2 2} + \frac{p_3(x)}{\alpha_3 4} + \cdots\cdots +}$$

HB separation theorems:

(1) If Y and Z are disjoint nonempty convex subsets of a TVS V, and Y is open, then there is a continuous linear functional p on V and a real number λ s.t.

$$\operatorname{Re} p(y) > \lambda \geq \operatorname{Re} p(z), \quad (y \in Y, \ z \in Z)$$

$V \to$ LCS space

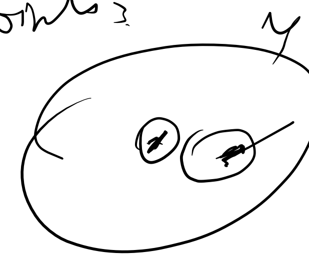
zero-functional is continuous.

Are there non-zero continuous functionals on V?

Are there closed co-dimension 1 subspaces of V?

If, further Z is open, then $\lambda > \operatorname{Re} \rho(z)$ for each z in Z.

Proof. Y consists entirely of internal points (as scalar multiplication and vector addition are continuous).



$\exists$ a function ρ on V s.t.

$$\operatorname{Re} \rho(y) > \lambda \geq \operatorname{Re} \rho(Z).$$

Let $|\alpha| = 1$, s.t. $\rho(\alpha x) = |\rho(x)|$.

Claim. If there is a nonempty open set G in V and a real number c s.t. $\mathrm{Re}\, \rho(x) < c$ whenever $x \in G$, then ρ is (uniformly) continuous on V.

$x_0 \in G$, $x_0 + V_0 \subseteq G$ for some balanced nbd of 0, V_0.

$x \in V_0$, $\rho(\alpha x) = |\rho(x)|$, $\boxed{|\alpha| = 1}$

$$c > \mathrm{Re}\, \rho(x_0 + \alpha x)$$
$$= \mathrm{Re}\, \rho(x_0) + |\rho(x)|$$

$$\Rightarrow \quad |\rho(x)| < \underbrace{\boxed{c - \mathrm{Re}\, \rho(x_0)}}_{(>0)} \quad \forall x \in V_0$$

$\Rightarrow |f(x)| < \varepsilon \quad \forall x \in \varepsilon b^{-1} V_b.$

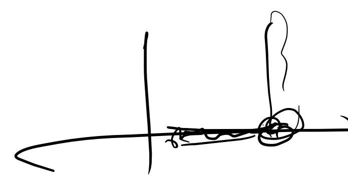
$\Rightarrow f$ is continuous at 0.

$\Rightarrow$ Use this result for $-f$ on Y.

$M[0,1] \to$ space of measurable functions on $[0,1]$ ~~odd~~ Borel a.e.-a.e. (μ - Lebesgue measure)

Topology of convergence in measure

$$d(f,g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\, d\mu$$



$\boxed{\mathcal{M}[0,1]}$

What does a nonempty convex $\boxed{\text{open}}$ set in $\mathcal{M}[0,1]$ look like?

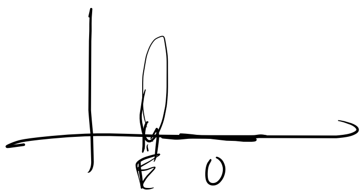
Claim: The only such set is $\mathcal{M}[0,1]$.

Proof: Assume ~~$\forall$~~ Y is another such set, and $Y \neq \emptyset \subset \mathcal{M}[0,1]$.

$$\operatorname{Re} f(y) > \lambda \geq \operatorname{Re} f(z)$$

$\mu(E) < \delta$ for E & measurable on $[0,1]$

$a\chi_E \in B(0, m\delta)$ $\forall a \in \mathbb{C}$



$d(a\chi_E, 0) \leq \mu(E) < \delta$

$\Rightarrow a\chi_E \in B(0, \delta)$

$\Rightarrow a\chi_E \in B(0, 1)$ $\forall a \in \mathbb{C}$

$f : M[0,1] \to \mathbb{C}$ is continu.

$\Rightarrow$ f is bounded on some ball around 0

$\Rightarrow$ $f(x) < \delta$ $\Rightarrow$ $f(ax) < \delta$ $\forall a \in \mathbb{C}$

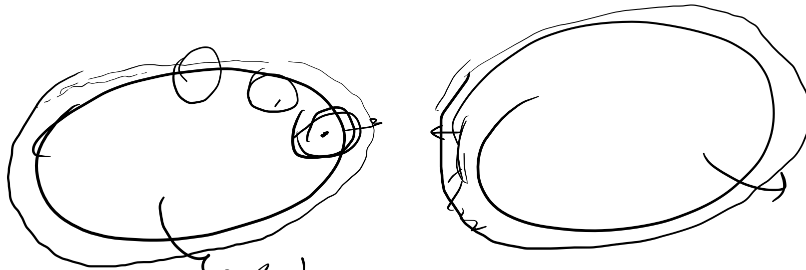
$\Rightarrow f(x) = 0 \quad \forall x \in M[0,1]$

$M[0,1]$ is not <u>locally convex</u>.

(2) If Y and Z are disjoint nonempty closed convex subsets of an LCS V, at least one of which is compact, then there are real numbers λ, μ such that

$$\operatorname{Re} f(y) \geq \lambda > \mu \geq \operatorname{Re} f(z) \quad (y \in Y,\ z \in Z)$$

Proof.



$(y+V_y) \cap Z = \phi$.

$\{y+\frac{1}{3}U_y\}$ is an open cover of Y.

$y_1, \dots, y_n$. $\boxed{\{y_i+\frac{1}{3}U_{y_i}\}}$ is an open cover of Y.

$U = \cap \frac{1}{3}U_{y_i}$

$Y+U$, $Z+U$ are open

$\{y+U : y \in Y\}$

$$(Y+U) \cap (Z+U) = \phi$$

$$y + v_1 = z + v_2$$

$$\Rightarrow z = y_1 + (y - y_1) + v_1 - v_2$$

$$\in y_1 + U + U + U$$

$$\Rightarrow z \in y_1 + U_{y_1}.$$

~~Contradiction~~ Contradiction!!!

COROLLARY – If x is a non-zero vector in an LCS then there is a continuous linear functional ρ on V such that $\rho(x) \neq 0$ || $x, y \in V$ $\exists\, \rho$, $\rho(x) \neq \rho(y)$]
$x \neq y$

Proof –, ~~Z~~ $Y \to$ singleton $\{x\}$.

$Z = \{0\}$!.

There is at least one closed codimension -1 subspace of V.

$\{|\rho| :\ \rho$ is continuous linear functional on $V\}$ is a separating family of seminorms!

($V \to$ space of continuous functionals.)

$f_p : V \longrightarrow \mathbb{R}$, p — continuous linear functional

$\cancel{p(x)} =$

$f_p(x) = p(x)$

$$\begin{tikzcd}
 & \mathbb{R}^\alpha \arrow[d, "\mathrm{proj}_p"] \\
V \arrow[ur, "f"] \arrow[r, "f_p"'] & \mathbb{R}
\end{tikzcd}$$

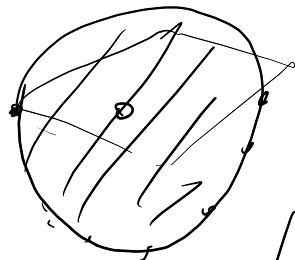
$\alpha =$ cardinality of ~~space~~ set of continuous linear functional.

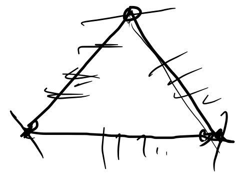
$f_p = \mathrm{proj}_p$ of f

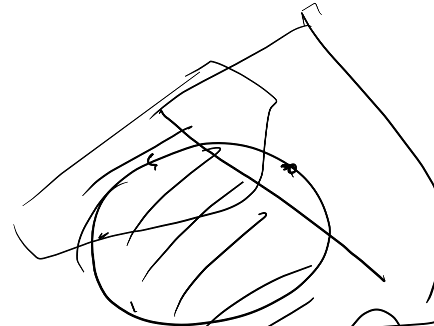
f is continuous from V to $\mathbb{R}^\alpha$, in the "product topology".

The evaluation map. is a topological embedding

since $\langle \rho \rangle$ is separating on V,







"closed" convex set

supporting hyperplane representation

external representation

$$A x \le b$$

$\vec{x} \in \mathbb{R}^n$