17/02/2022 Lecture - 9

$\mathbb{K}$-spaces

n-D subspaces of $\boxed{V^{**}}$ $\longleftrightarrow$ Codimension - n subspace of V

If V is a TVS, then it has a base of 0 consisting of balanced ~~convex~~ "convex" set nbhds of 0

If x is close to $\boxed{x_0}$, $f(x)$ is close to $\boxed{f(x_0)}$.

$\downarrow$

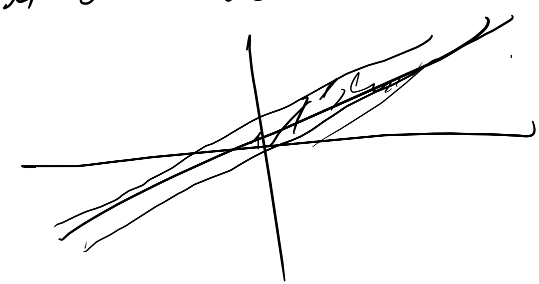
$f(x_0)$

Continuity is defined at a point of X.

Uniform continuity has to be defined on all of X simultaneously,

(x, y) is close to $(x_0, x_0) \in \mathrm{diag}(X \times X)$

then $(f(x), f(y))$ is close to the diag $(X \times X)$



$(f(x_i), f(y_i))$

Proposition: Suppose V, W are TVS's over the same scaler field.

If $x_0 \in V$ and T is continuous at x_0, then T is uniformly continuous on V.

Proof: $\quad \mathcal{E}_V(U) = \{ (x,y) \in V \times V ; \; x - y \in U \}$

U is a nbhd of 0 in V.

(If (x, y) is close to the diagonal in V.
(Tx, Ty) is close to the diagonal of W.)

$Tx - Ty \in U_W$ → nbd. of 0 in W.

$(x,y) \in$

$\exists\ U_V$ nbd. of 0 in V s.t. whenever

$\boxed{x-y} \in U_V$, then $Tx - Ty \in U_W$.

$T(x-y) \in U_W$.

$V \times V$

$(x+x_0) - (y+x_0)$

(x_0, x_0)

$\boxed{(x,y) - (x_0, x_0)}$ does

$(V \times V)$

$(Tx, Ty) - (Tx_0, Tx_0)$

$\to T(x+x_0) - T(y+x_0) \in U$

This follows from continuity of T at 0

<u>Corollary</u>: If V, W are TVS's, $\boxed{W \text{ is complete}}$

V_0 → everywhere dense subspace of V and

$T_0 : V_0 \to W$ is a continuous linear operator.

then T_0 extends uniquely to a continuous linear operator $T: V \to W$.

$$T : V \to W$$

Any ~~linear~~ continuous linear map takes Cauchy nets to Cauchy nets.

$$\boxed{\tilde{T} : \tilde{V} \longrightarrow \tilde{W}}$$

$\Big|\ T(\lim x_i) = \boxed{\lim_i (T x_i)}$

$(x_i) \longrightarrow x$ $\quad (T x_i)$ is also

$Tx = \lim_i T x_i$

Proof.

$$g(x,y) = T(ax+by) - aTx + bTy$$

$$\boxed{Tx + Ty - T(x+y) = 0} \quad \forall x,y \in V_0$$

g is a continuous function, $\forall x, y \in V_0$.

$$\tilde{g}(x,y) = \tilde{T}(ax+by) - a\tilde{T}x + b\tilde{T}y$$

is a continuous function on $V \times V$.

$\tilde{g}(x,y) = 0$ on the dense subspace $V_0 \times V_0$.

$$\tilde{T}(ax+by) = a\tilde{T}x + b\tilde{T}y.$$

~~Proof~~

Proposition: A linear functional ρ on a TVS V is continuous if and only if its nullspace $\rho^{-1}(\{0\})$ is closed in V.

Proof: $\rho^{-1}(\{0\})$ is closed in V if ρ is continuous ($\because$ $\{0\}$ is closed as V is T_2.)

Assume $\rho^{-1}(\{0\})$ is closed. If $\rho^{-1}\{0\} = V \Rightarrow \rho \equiv 0$
$\Rightarrow \rho$ is continuous

Let $x_0 \in V$ s.t. $\rho(x_0) = 1$.

There is a $\boxed{\text{balanced}}$ U_0 of 0 in V s.t.

$$(x_0 + U_0) \cap \boxed{\rho^{-1}(\{0\})} = \phi.$$

Let, if possible, $x \in U_0$ and $\boxed{|f(x)| \geq 1}$.

and $a \in \mathbb{C}$, s.t. $f(ax) = -1$, $|a| \leq 1$

$ax \in U_0$, $x_0 + ax \in \boxed{x_0 + U_0}$

$f(x_0 + ax) = f(x_0) + f(ax) = 1 + -1 = 0$

$x_0 + ax \in f^{-1}(\alpha_0)$

Thus, $\boxed{|f(x)| < 1 \quad \forall x \in \boxed{U_0}}$

$\Rightarrow f$ is continuous.

$f : V \to \mathbb{C}$.

U

$f(U) \subseteq B(0, \varepsilon)$

$\Leftarrow f(\varepsilon U) \subseteq B(0, 1)$

Prop^n — In a TVS, a subspace of codimension one is either <u>dense</u> or <u>closed</u>.

<u>Proof</u>. $W \subseteq V$.

$$W \subseteq \overline{W} \subseteq V$$

$\Rightarrow$ $W = \overline{W}$ (closed)

or, $\overline{W} = V$ (W is dense in V).

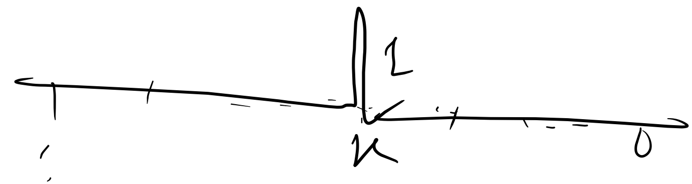
<u>Example</u>: $l_{00}(\mathbb{N}) \longrightarrow$ space of complex sequences which are eventually 0.

$\rho(\delta_n) = n.$

$\downarrow$

$$\delta_n(k) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

$\mathrm{Span}\{\delta_n : n \in \mathbb{N}\}$



$= \ell_{00}(\mathbb{N}),$

$$\left[\ \rho\left(\sum_{i=1}^{n} a_i \delta_i\right) = \sum_{i=1}^{n} a_i \cdot i\right.$$

$$\rho\left(\frac{1}{n} \cdot \delta_n\right) = 1.$$

$\frac{\delta_n}{n} \to 0$ as $n \to \infty$.

$\rho\left(\frac{\delta_n}{n}\right) = 1 \longrightarrow 1$ as $n \to \infty$

nullspace of ρ is dense in $l_\infty(\mathbb{N})$.

HB separation → ~~closed~~ convex sets separated by "closed" hyperplanes

HB extension → "extensions" of continuous linear functionals.

$$\rho(x) \leq |\rho(x)|$$

Locally Convex Spaces (LCS)

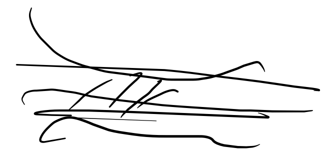
We have that every TVS has a base of nbhds of 0 that are balanced.

A TVS is said to be **locally convex** if it has a base of [balanced convex nbhds around 0.]

Example - $\mathbb{K}^n$ are locally convex spaces.

(with the product topology)

Locally convex topology on V, is a topology which makes V an LCS.



Example $\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$, $C_0(\mathbb{R}) \ni$

$$B(0,\varepsilon) = \{ g \in C_0(\mathbb{R}) : |g(x)| < \varepsilon(x) \ \forall x \in \mathbb{R} \}$$

$\boxed{\text{LCS}}$ but not $\boxed{\text{metrizable}}$

$$|\lambda g_1(x) + (1-\lambda) g_2(x)| \le \lambda |g_1(x)| + (1-\lambda)|g_2(x)| < \varepsilon(x)$$

$\boxed{\text{separating family of seminorms on } V}$ (Γ) $\longleftrightarrow$ $\boxed{\text{locally convex topologies on } V}$

for every $x \in V$,

$\exists\, p \in \Gamma$ s.t. $p(x) \neq 0$

$x, y \in V$

$\exists\, p \in \Gamma$ s.t. $p(x-y) \neq 0$

$|p(x) - p(y)|$

$p(x-y)$

Given Γ, let τ be the coarsest topology on V s.t. $p \in \Gamma$ is continuous.

Theorem: Suppose V is a $\mathbb{K}$-space, and Γ is a family of seminorms on V that is *separating* - if $x \in V$ and $x \neq 0$, $\exists$ $p \in \Gamma$ s.t. $p(x) \neq 0$.

Then there is a locally convex topology on V in which for each $x_0 \in V$, the family of sets

$$V(x_0; p_1, \ldots, p_m; \varepsilon)$$

$$= \{x \in V ; \ p_j(x - x_0) < \varepsilon, \ j = 1, \ldots, m\}$$

(where $\varepsilon > 0$, $p_1, \ldots, p_m \in \Gamma$) is a base of nbds of x_0.

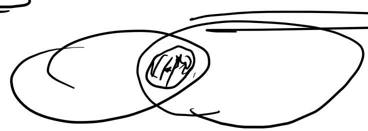
With this topology, each of the seminorms on Γ is continuous.

Moreover, every locally convex topology arises in this manner from some separating family of seminorms on V.

Proof. ($\Rightarrow$). $\mathcal{B}$ is a base. $B_1, B_2 \in \mathcal{B}$, $\exists B_3 \in \mathcal{B}$ s.t. $B_3 \subseteq \boxed{B_1 \cap B_2}$

Step I. $\{V(x_0; p_1, \dots, p_m; \varepsilon) : x_0 \in V, p_1, \dots, p_m \in \Gamma, \varepsilon > 0\}$ is a base for a topology τ on V.

Let $\boxed{x_0} \in V(y_0; p_1, \dots, p_m; \delta) \cap V(z_0; q_1, \dots, q_n; \eta)$.



$$0 < \boxed{\varepsilon} < \inf_{\substack{j=1,\dots,m \\ k=1,\dots,n}} \{ \underbrace{\delta - p_j(x_0 - y_0)}_{>0}, \underbrace{\eta - q_k(x_0 - z_0)}_{>0} \}$$

$$x \in V(x_0; p_1, \dots, p_m, q_1, \dots, q_n; \varepsilon)$$

$$\subseteq \boxed{V(y_0; p_1, \dots, p_m; \delta)} \cap \boxed{V(z_0; q_1, \dots, q_n; \eta)}$$

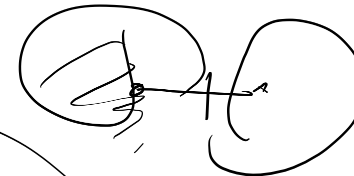
$$p_1(x - x_0) < \varepsilon < \delta - p_1(x_0 - y_0)$$

$$\Rightarrow p_1(x - y_0) \leq p_1(x - x_0) + p_1(x_0 - y_0) < \delta$$

$$q_1(x - z_0) \leq q_1(x - x_0) + q_1(x_0 - z_0) < \eta.$$

Step II – τ is a Hausdorff topology

Let $x_0, y_0 \in V$ s.t. $x_0 \neq y_0$.



Let $p \in \Gamma$ s.t. $p(x_0 - y_0) > 0$

Choose ε s.t. $0 < \varepsilon < \frac{1}{2} p(x_0 - y_0)$

$$V(x_0; p; \varepsilon) \cap V(y_0; p; \varepsilon) = \emptyset.$$

Step III. $(x,y) \mapsto x+y$, $(a,x) \mapsto ax$

is continuous w.r.t. τ (that is, (V,τ) is a TVS).

$x_0 + V(0; p_1, \dots, p_m; \varepsilon)$

$$\boxed{V(x_0; p_1, \dots, p_m; \tfrac{\varepsilon}{2})} + V(y_0; p_1, \dots, p_m; \tfrac{\varepsilon}{2})$$

$$\subseteq V(x_0 + y_0; p_1, \dots, p_m; \varepsilon) \subseteq U$$

$$B(a_0, \delta_0) \cdot V(x_0; p_1, \dots, p_m; \delta)$$

$$\subseteq V(a_0 x_0; p_1, \dots, p_m; \varepsilon),$$