15/02/2022

Lecture-8

K — convex subset of a $\mathbb{K}$-space and 0 is an internal point.

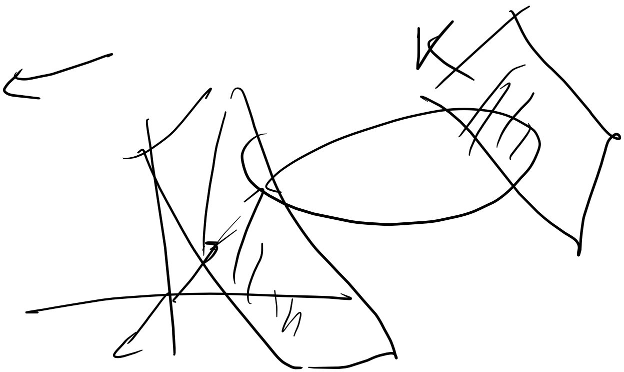
$p(x) = \inf \{ c \in \mathbb{R} : c > 0, x \in cK \}$.

If K consists entirely of internal points, then $K = \{ x \in K : \boxed{p(x) < 1} \}$. $\to h_{K^*}(x) < 1$.

nonempty closed convex subsets of $\mathbb{R}^n$.

sublinear functional

h_K ←



$p = h_{K^*}$ ($K^* \to$ polar body of K)

$h_K = \sup \{ \langle x, u \rangle ; x \in K \}$. $K \longleftrightarrow K^*$

(one-to-one correspondence on the set of convex bodies
↓
nonempty convex subsets with nonempty interior)

$K^* = \{ y \in \mathbb{R}^n ; \langle x, y \rangle \leq 1 \ \forall x \in K \}$

$$\boxed{K^* = \{ y \in \mathbb{R}^n ; h_K(y) \leq 1 \}}$$

$(K^*)^* = K$

n-D subspace of $V^{\#}$ $\longleftrightarrow$ Codimension-n subspace of V

Hahn-Banach separation theorem is about "separating" convex sets using hyperplanes

convex sets $\downarrow$ sublinear functionals.

hyperplanes $\downarrow$ ρ

Hahn - Banach $\boxed{\text{extension}}$ theorem w.r.t. sublinear functionals (on real vector spaces)

If p is a sublinear functional on a real vector space V, while ρ_0 is a linear functional on a linear subspace V_0 of V, and

$$\rho_0(y) \leq p(y), \quad (y \in V_0).$$

then there is a linear functional ρ on V such that $\boxed{\rho(x) \leq p(x) \quad \text{for } x \in V}$

and $\rho(y) = \rho_0(y)$ for $y \in V_0$.

[$p : V \to \mathbb{R}$ is a convex function.

equivalently, $\mathrm{epi}(p) = \{ (x, p(x)) \in V \times \mathbb{R} ;$
$x \in V \}$.]

$p|_{V_0}$ is a convex function.

and p_0 is an affine (linear) underestimator.

Proof: Consider $V \times \mathbb{R}$ as a real-vector space.

$$K := \{(x, r) \in V \times \mathbb{R} : p(x) < r\}.$$

~~(...)~~

$K \longrightarrow$ nonempty, convex and consists entirely of internal points.

$$\boxed{p(x_1) < r_1} \qquad p(\lambda x_1 + (1-\lambda) x_2)$$

$$\boxed{p(x_2) < r_2} \qquad < \lambda r_1 + (1-\lambda) r_2.$$

$(x_1, r_1) \in K$, $(x_2, r_2) \in K$

$$(\lambda x_1 + (1-\lambda) x_2, \lambda r_1 + (1-\lambda) r_2)$$

$$\in K.$$

$(x, r) \in K$, $\qquad (y, \alpha) \in V \times \mathbb{R}$.

$\Downarrow$

$p(x) < r$

& Want to find $\varepsilon > 0$ such that $p(x + \varepsilon y) < r + \varepsilon\alpha$.

$$p(x+\varepsilon y) \leq p(x) + \varepsilon p(y) \leq r + \varepsilon\alpha$$

$$\Rightarrow \quad \varepsilon(p(y) - \alpha) < \boxed{r - p(x)}$$

Case I. If $p(y) \leq \alpha$, then any $\varepsilon > 0$ suff....

Case II, If $p(y) > \alpha$, chose $\varepsilon < \dfrac{r - p(x)}{p(y) - \alpha}$

$W = \{ (\cancel{p_0(y)}, y, p_0(y)) ; y \in V_0 \}$.

$W \to$ linear subspace of $\mathbb{R} \times V$.

(hence, convex).

$$\boxed{K \cap W = \phi.}$$

~~$x \in W \Rightarrow$~~ $(x, y) \in W \Rightarrow p_0(x) = y$

$\Rightarrow y \leq p(x)$.

There is a linear functional σ on $V \times \mathbb{R}$ and $\lambda \in \mathbb{R}$ s.t.

$$\boxed{\sigma(v) > \boxed{\lambda} \geq \sigma(w)} \quad (v \in K, w \in W).$$

If $\omega \in W$, then $a\omega \in W$. $\forall a \in \mathbb{R}$.

$\Rightarrow \quad a\sigma(\omega) \leq \lambda$ for all $a \in \mathbb{R}$.

$\Rightarrow$ (i) $\boxed{\sigma(\omega) = 0 \quad \forall \omega \in W}$

(ii) $\lambda \geq 0$.

$\cancel{(\vec{1}, 0) \in K}$ $(\vec{0}, 1) \in K \Rightarrow \sigma((\vec{0}, 1))$

$> \lambda \geq 0$

WLOG, assume $\sigma((\vec{0}, 1)) = 1$.

$\rho(x) = -\sigma((\vec{x}, 0))$ defines a linear functional on V.

$$\sigma((\vec{x}, r)) = \sigma(\underline{(\vec{x}, 0) + (\vec{0}, r)})$$

$$= \sigma((\vec{x}, 0)) + \sigma((\vec{0}, r))$$

$$= -\rho(\vec{x}) + r > 0$$

where $(\vec{x}, r) \in K$.

$$\Rightarrow \quad \rho(\vec{x}) < r$$

whenever $p(\vec{x}) < r$

$$\Rightarrow \boxed{\rho(\vec{x}) \leq p(\vec{x}) \quad \forall\ \vec{x} \in V.}$$

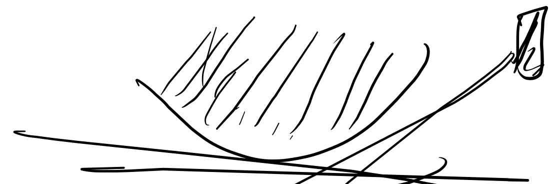
$\rho_0(\vec{y}) - \rho(\vec{y})$

$= \sigma\left(\boxed{(\vec{y}, \rho_0(\vec{y}))}\right) \in W$

$= \sigma((\vec{y}, 0)) + \sigma((0, \rho_0(\vec{y})))$

$= 0$

$\Rightarrow \rho(\vec{y}) = \rho_0(\vec{y}) \quad \forall \vec{y} \in V_0.$



$p = \max_{\boxed{i \in I}} \langle \rho_i h$ linear funct.

$p(x) \le \lambda \iff \rho_i(x) \le \lambda \quad \forall i,$

Hahn-Banach extension theorem w.r.t. seminorms

If p is a semi-norm on a $\mathbb{K}$-space V

and ρ_0 is a $\mathbb{K}$-linear functional on a

linear subspace V_0 of V, and

$$|\rho_0(y)| \leq p(y) \quad \text{for } y \in V_0,$$

$|\rho_0(x+y)|$

$= |\rho_0(x) + \rho_0(y)|$

$\leq |\rho_0(x)| + |\rho_0(y)|$

then there is a linear functional ρ on V

such that

$$|\rho(x)| \leq p(x) \quad \text{for } x \in V$$

and $\rho(y) = \rho_0(y)$, for $y \in V_0$.

Proof: $\underline{\mathbb{K} = \mathbb{R}}$.

p is a sublinear functional.

By HB extension thm, we have a linear functional ρ on V s.t.

$$\rho(y) = \rho_0(y) \quad \text{and} \quad \underline{\rho(x) \le p(x)} \quad \forall x \in V,$$

$$\Rightarrow \quad \rho(-x) \le p(-x) \quad \forall x \in V,$$

$$= p(x)$$

~~semi~~ as p is a seminorm.

$$\Rightarrow \quad -p(x) \le \rho(x) \quad \forall x \in V$$

$$|\rho(x)| \le p(x).$$

$\underline{K = \mathbb{C}}$ $\to$ $\rho_0(y) = \sigma_0(y) - i\,\sigma_0(iy)$

$\sigma_0 := \mathrm{Re}\,\rho_0$ is a $\mathbb{R}$-linear functional on $V_{\mathbb{R}}$.

$$\sigma_0(y) \leq p(y) \quad \forall y \in V_0$$

$$(\sigma_0(y) \leq |\rho_0(y)|)$$

There is a real-linear functional σ on $V_{\mathbb{R}}$ such that $\sigma(y) = \sigma_0(y)$, $y \in V_0$

and $\boxed{\sigma(x) \leq p(x)}$, $x \in V_{\mathbb{R}}$.

Let $\rho(x) := \sigma(y) - i\,\sigma(iy)$.

$\rho(y) = \rho_0(y) \quad \forall y \in V_0.$

Let $a \in \mathbb{C}$, s.t. $|a| = 1$, $|f(x)| = a f(x)$.

$$|f(x)| = f(ax) = \operatorname{Re} f(ax)$$

$$= \sigma(ax) \le p(ax)$$

$$= |a|\, p(x)$$

$$= p(x), \ \forall x \in V.$$

$$\Rightarrow \quad |f(x)| \le p(x), \quad \forall x \in V.$$

$\blacksquare$

$\forall$ V^* has plenty of "continuous" linear functionals.

(1) Continuity of functionals

(2) Extension of functionals

(3) Concrete characterization of space of functionals.

$x^n \longmapsto \int_0^1 x^n \, dx,$

$$c_n = \int_0^1 \boxed{x^n} \, d\mu(x)$$

$(c_1 \ldots c_n \ldots c_n \ldots)$

Orals. Does it come from a measure;

$(C[0,1])^* \equiv$ complete Radon measures on)

Topological vector spaces (Want to talk about continuous linear maps)

V ($\mathbb{K}$-space) with a **Hausdorff** topology

such that

$$V \times V \longrightarrow V \quad \text{(a vector addition)}$$
$$(x, y) \longmapsto x + y$$

$$\mathbb{K} \times V \to V \quad \text{(scalar multiplication)}$$
$$(a, x) \mapsto ax$$

are continuous maps.

<u>Example</u> (i) $\mathbb{K}^n$ with the product topology.

(ii) $C[0,1]$ with sup norm is a complete metric space. Consider $C[0,1]$ with the supnorm topology.

* Complex TVS is a real TVS by restricting scalar multiplication to $\mathbb{R}$.

* ~~A~~ A vector subspace of a TVS is ~~a~~ again a TVS with the relative (or subspace) topology.

Lemma.

(1) If W is a vector subspace of a TVS V, then $\overline{W}$ is also a vector subspace.

(2) If a subset W of a TVS V is balanced (or convex), then the same holds for $\overline{W}$.

(3) An open set G in V consists entirely of internal points.

Proof.

$$\overline{W} = \bigcap_{\substack{\cancel{a \in V \cup \{0\}} \\ V \text{ a ball nbhd of } 0.}} (V + W)$$

$W \times W$ is dense in $\overline{W \times W}$

$(x, y) \in \overline{W} \times \overline{W}$.

$ax + by \in \overline{W}$.

and nets

$(x_i, y_i) \to (x, y)$

$ax_i + by_i \to$

$W \times W$ is dense in $\overline{W} \times \overline{W}$.

$\downarrow$

$\cancel{A}$ V

$$\begin{array}{ccc} (x_i, y_i) & \longrightarrow & (x, y) \\ \downarrow f & & \\ x_i + y_i & \longrightarrow & x + y \\ \in W & & \in \overline{W}. \end{array}$$

$\overline{W}$ is closed under ~~a~~ vector addition.

(2) W is balanced $\quad aW \subseteq W \quad \forall\, |a| \leq 1$

$$\boxed{\begin{array}{l} x_i \in W \longrightarrow x \in \overline{W} \\ a x_i \in W \xrightarrow[\text{of } a \cdot h]{\text{continuous}} a x \in \overline{W}. \end{array}} \Rightarrow \overline{W} \text{ is balanced.}$$

* U is a nbd. of 0 iff $x_0 + V$ is a nbd. of x_0 ($U \to V$, $x \longrightarrow x_0 + x$ is a homeomorphism)

Translation by a fixed vector x_0 is a homeomorphism)

The topology on a TVS is translation-invariant.

[Topology of a TVS is completely determined by a base of nbd.-s of 0].

Lemma: Every nbd. of 0 in a TVS V contains a balanced nbd. of 0.

Proof: Let U be a nbd. of 0.

($\Rightarrow$ λU is a nbd. of 0 for $\lambda \neq 0$

($\because$ $x \longmapsto \lambda x$ is a homeomorphism for $\lambda \neq 0$))

$\exists$ a nbd. U_1 of 0 and $\varepsilon > 0$

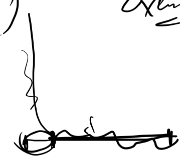
s.t. $ax \in U$ whenever $x \in U_1$ and $\boxed{|a| \leq \varepsilon}$.

$\mathbb{K} \times V \to V$

$\Rightarrow \bigcup_{\boxed{0 < \lambda \le 1}} \{\lambda U\}$ is a balanced open subset of U.

[Thus, TVS's have a base (at 0), given by balanced nbhds of 0.]

Convex



Example: $\mathcal{M}([0,1])$ — measurable functions on $[0,1]$ identifying μ-a.e. equal functions

$$d(f,g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\, d\mu$$

μ — Lebesgue measure.

d is a metric on $M([0,1])$

What does a nonempty convex open set in $M[0,1]$ look like?

There is only one such set, that is, $M([0,1])$ itself.

$f_n^2 \to f^2$.

$(f_n+g)^2 \to f^2+g^2+2gf$.

$x_{n_i} \to x$ in X.

~~eventua~~ given a nbd. of X, x_i is eventually ~~contained~~ completely in that nbd.

Want to say: $(x_i)_{i \in I}$ is Cauchy in X

$(x, y) \in X \times X$

$(\forall m, n \geq N_\varepsilon \; d(x_m, x_n) < \varepsilon)$

Uniform structure on X:

topology helps us define continuity, convergence.

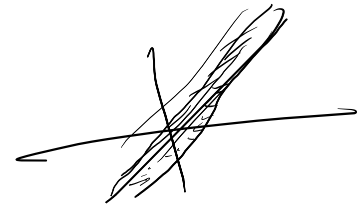
"uniform structure" helps us define uniform continuity

uniform convergence.

(X, Φ) → uniform structure

$\Phi \subseteq P(X \times X)$.

elements of Φ are <u>entourages</u>.



(1) $U \in \Phi \Rightarrow \Delta = \{(x,x) : X \times X\} \subseteq U$.

(x is uniformly close to x.

$X \to X$

$x \mapsto x$ is uniformly continuous)

(2) If $U \in \Phi$, and

$U \subseteq V \subseteq X \times X$, $V \in \Phi$,

then $V \in \Phi$

(3) $U, V \in \Phi$, $U \cap V \in \Phi$.

(4) $U \in \Phi$, $\exists V \in \Phi$ s.t. $(x,y) \in V$, $(y,z) \in V$

$\Rightarrow (x,z) \in U$. (Halving of U)

(5) $U \in \Phi \Rightarrow U^{-1} := \{(y, x) : (x, y) \in U\}$

(symmetric)

(X, τ, Φ) - uniform topological space.

Notion of Cauchy net:

(x_i) is Cauchy if for any $\forall U \in \Phi$,

$\exists\, j \in I$ s.t. $\forall\, i_1, i_2 \geq j$ $(x_{i_1}, x_{i_2}) \in U$.

Ex. Given an open nbd. U of $0 \in V$,

$$\varepsilon(U) = \{(x,y) : y - x \in U\} \subseteq V \times V$$

defines a uniform structure on V.

$$(V, \tau, \varepsilon).$$

$$\varepsilon(U) \cap \varepsilon(U') \subseteq \varepsilon(U'')$$

for some U'' open nbd. of 0.

$U'' = U \cap U'$.

Assume ~~that U is balanced~~

$V \times V \xrightarrow{f} V$

U

$f^{-1}(\cdot)$

$U_1 \times U_2 \subseteq f^{-1}(U)$

$\boxed{U_1 \cap U_2} \times U_1 \cap U_2$

f

$\varepsilon(U_1 \times U_2) \subseteq U$

$f : X \to Y$ (X, Y are uniform spaces)

f is said to be <u>uniformly continuous</u> if

given entourage E on Y, $\exists$ entourage D in X s.t. $\forall\, a, b \in X$

$$(a,b) \in D \Rightarrow (f(a), f(b)) \in E$$

[(a,b) are D-close $\Rightarrow$ $f(a), f(b)$ are E-close]

Proposition: Suppose V, W are TVS's over same $\mathbb{K}$. $T: V \to W$ is a linear mapping.

~~(B)~~ If $x_0 \in V$ and T is continuous at x_0, then T is uniformly continuous on V.

Proof: T is uniformly continuous if for G an open nbd. of 0 in W, $\exists$ F an open nbhd of 0 in V, s.t. whenever $(x, y) \in \mathcal{E}(F)$, $(Tx, Ty) \in \mathcal{E}(G)$.

$(x - y + x_0, x_0) \in \mathcal{E}(F)$ $\quad$ $(Tx - Ty + Tx_0, Tx_0)$

$x_0 + (x - y) \in x_0 + F$ $\quad$ $Tx_0 + Tx - Ty \in Tx_0 + G$

(1-D sp of continuous linear functionals on V^*

$\longleftrightarrow$ closed linear subspace of codimension 1 in V)

$W \hookrightarrow V$ of codimension.

$V \to V/W$

$W \subseteq \overline{W} \subset V$

$\boxed{\overline{W} = V}$

$\overline{W} = W$

"Locally convex sp"

$\longleftrightarrow$ separating family of seminorms

V with trivial topology.