10/02/2022

Lecture-7

Theorem (Hahn-Banach separation theorem for complex vector spaces)

Let Y, Z be non-empty disjoint convex subsets of a complex vector space V.

(1) If one of Y, Z has an internal point, then there is a non-zero (complex) functional ρ on V and a real number λ, such that

$$\operatorname{Re}\rho(y) \geq \lambda \geq \operatorname{Re}\rho(z), \quad (y \in Y,\ z \in Z)$$

(2) If Y consists entirely of internal points,

$$\operatorname{Re}\rho(y) > \lambda \quad \forall\, y \in Y.$$

$\sigma \longrightarrow x + s$

$\underline{\sigma(x) - i\sigma(ix)} \to \rho$

$\operatorname{Re}\rho = \sigma$

(3) If Y, Z consist entirely of internal points, then $\operatorname{Re} \rho(y) > \lambda > \operatorname{Re} \rho(z)$, $\forall y \in Y, z \in Z$.

Proof: Consider Y, Z as subsets of $V_{\mathbb{R}}$.
There is a real-linear functional σ with the specified properties.
By previous lemma, $\exists \rho$ s.t. $\sigma = \operatorname{Re} \rho$.

$\blacksquare$

n-D subspace of $V^\#$ $\longleftrightarrow$ Codimension-n subspaces of V.

Let V be a $\mathbb{K}$-vector space.

(1) Sublinear functional on V; $p : V \to \mathbb{R}$ such that $p(x+y) \le p(x) + p(y)$, $p(ax) = a\,p(x)$

$\forall\, x, y \in V$ and $a \ge 0$.

($p_1, p_2 \in V^\#$ real. linear functionals on V,

$\max\{p_1, p_2\}$ is a sublinear functional.)

$\longleftrightarrow$ "convex sets" in V.

$(V \cong \mathbb{R}^n)$

(2) Seminorm on V $\quad p: V \to \mathbb{R}$ such that

p is a sublinear functional, and $p(ax) = |a|\, p(x)$ $\forall a \in \mathbb{R}$.

$\Longleftrightarrow$ "origin symmetric" convex sets (in $V \cong \mathbb{R}^n$)

$$2p(x) = p(x) + p(-x) \geq p(0) = 0$$

$$\Rightarrow p(x) \geq 0.$$

(3) Norm on V $\quad p: V \to \mathbb{R}$ such that p is

a seminorm and $p(x) > 0$ whenever $x \in V$, $x \neq 0$.

$\Longleftrightarrow$ "origin-symmetric" convex sets with 0 in the interior ($V \cong \mathbb{R}^n$)

Example: $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$V = \mathbb{K}^n$.

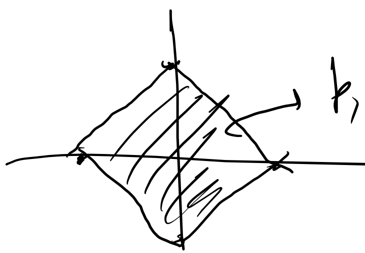
$$\boxed{p_1}((a_1, \dots, a_n)) = |a_1| + \dots + |a_n| \quad (\ell^1\text{-norm on } \mathbb{K}^n)$$

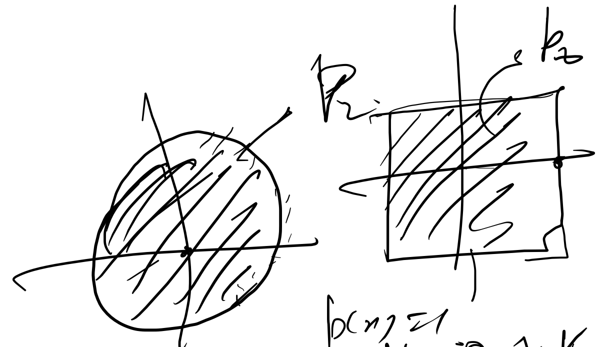
$$p_k((a_1, \dots, a_n)) = \left(|a_1|^k + \dots + |a_n|^k\right)^{1/k} \quad (\ell^k\text{-norm on } \mathbb{K}^n)$$

$$p_\infty((a_1, \dots, a_n)) = \max\{|a_1|, \dots, |a_n|\}.$$

$\mathbb{K} = \mathbb{R}$, 'unit ball in the norm' is the corresponding geometric object.

$n = 2$.

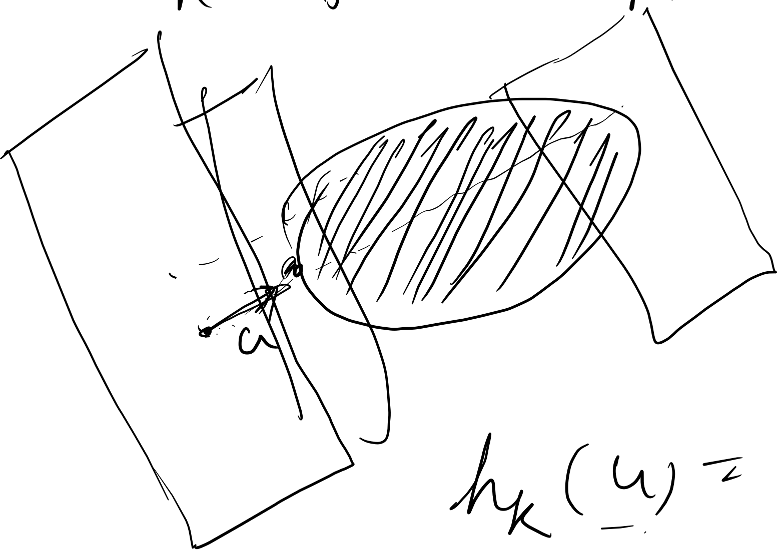




$p(x) = 1$
$\forall x \in \partial K$

Support function of a nonempty closed convex set in $\mathbb{R}^n$

$K \subseteq \mathbb{R}^n$ nonempty closed convex set in $\mathbb{R}^n$,



$$h_K : S^{n-1} \to \mathbb{R}$$

offset.

$$h_K(\underline{u}) = \sup \{ \langle x, u \rangle : x \in K \}$$

for $u \in \mathbb{R}^n$

h_K — support function of K

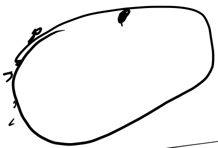
$$h_K(\lambda u) = \lambda h_K(u), \quad \lambda \geq 0.$$

$$h_K(u+v) = \sup\{\langle u+v, x\rangle ; x \in K\}$$

$$\langle u, x\rangle + \langle v, x\rangle ; x \in K$$

$$\le \sup\{\langle u, x\rangle ; x \in K\} + \sup\{\langle v, x\rangle ; x \in K\}$$

$$= h_K(u) + h_K(v).$$

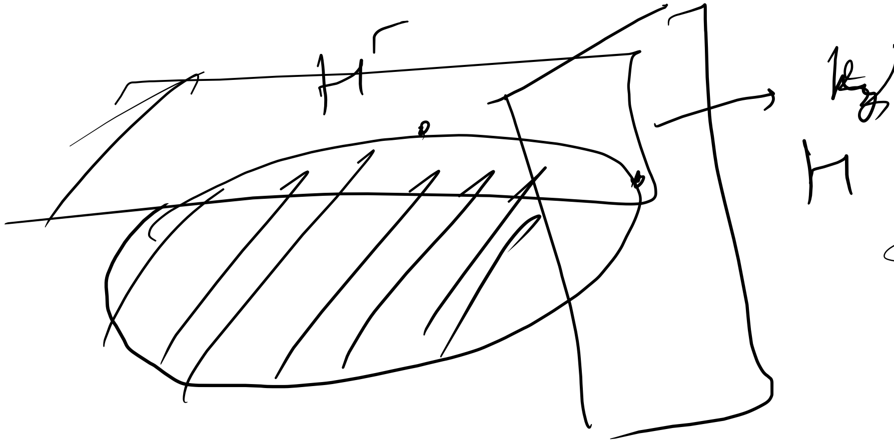


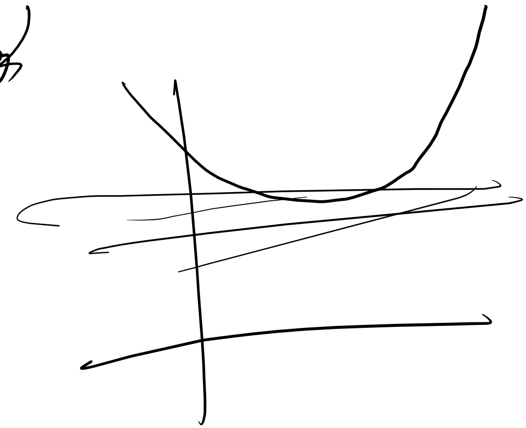
vertices, (extreme pts

extreme rays

subspaces

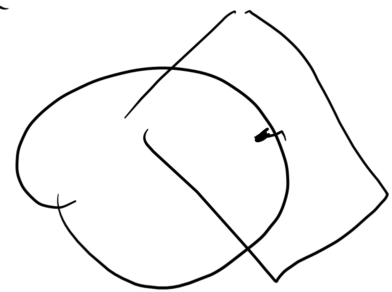
Support function of K is a sublinear functional on $\mathbb{R}^n$.





$K = \bigcap \overline{H}^{-}_{y,\alpha}$

Every closed convex set in $\mathbb{R}^n$ is the intersection of closed halfspaces.



$h_K(u) - h_K(-u)$

= width of K in the direction of u.

h_K is uniquely associated with K.

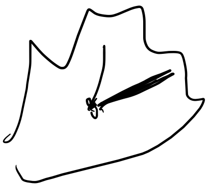
Given h_K it is easy to recover K.

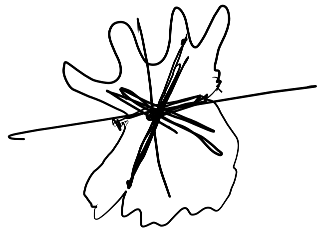
Theorem: If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique nonempty compact convex set K with support function $f = h_K$.

sublinear functionals on $\mathbb{R}^n$ $\longleftrightarrow$ nonempty compact convex sets of $\mathbb{R}^n$

Balanced set in V ($\mathbb{K}$-space)

A subset Y of V is said to be *balanced* if $ay \in Y$ whenever $y \in Y$, $a \in \mathbb{K}$, $|a| \leq 1$.





When $\mathbb{K} = \mathbb{R}$,

"balanced" $\Longleftrightarrow$ "origin-symmetric" ~~and~~ star shaped body containing 0.

$a \in D$

Observation :

(1) If p is a sublinear functional on V,

$$V_p = \{ x \in V : p(x) < 1 \}$$

- V_p is convex.

$x, y \in V_p$ $\quad \lambda \in [0,1]$

$$p(\cancel{x+} \lambda x + (1-\lambda) y)$$
$$\leq p(\lambda x) + p((1-\lambda) y)$$
$$\leq \lambda p(x) + (1-\lambda) p(y)$$
$$< \lambda + (1-\lambda) = 1.$$

- V_p contains 0.

- V_p consists entirely of internal points.

$x \in V_p$, $y \in V$.

$$p(x + \varepsilon y) \leq p(x) + \varepsilon p(y) < 1$$

$$\Rightarrow \varepsilon < (1 - p(x)) / p(y)$$

(2) V_p is balanced if p is a semi-norm.

$p(ax) = |a|\, p(x)$

$a \in \mathbb{C}$.

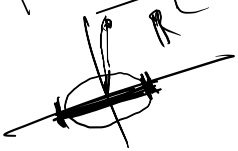
If $p(x) < 1$, and $\boxed{|a| \leq 1}$,

$\Rightarrow \quad p(ax) < 1$

$\Rightarrow \quad ax \in V_p$.

Proposition: Suppose that K is a convex subset of a $\mathbb{K}$-space V and 0 is an internal point of V. Let, $p(x) := \inf\{c \in \mathbb{R} : c > 0, x \in cK\}$.

$\boxed{\Rightarrow K \text{ is absorbing.}}$



(1) p defines a sublinear functional on V,

(2) If K consists entirely of internal points,

$$K = \{x \in V : p(x) < 1\}.$$

(3) If K is balanced, p is a seminorm.

Proof:- (1) Let $x \in K$,

$\frac{1}{c} x \in K$ for sufficiently large c

(as 0 is an internal point).

$\Rightarrow$ $x \in cK$ for " " ".

$\Rightarrow$ $p(x)$ is a non-negative real number

($p(0 \cdot x) = p(0) = 0$
$= 0 \cdot p(x)$)

$a > 0$,

$x \in cV \Leftrightarrow ax \in \boxed{ac} V$

$p(ax) = \inf \{ c \in \mathbb{R} : c > 0, ax \in cK \}$.

$= \inf \{ c \in \mathbb{R} : c > 0, x \in \frac{c}{a} K$

$= \inf \{ ac : c > 0, x \in K \} = a p(x)$.

Let $x, y \in V$, $\varepsilon > 0$,

$$0 < b < p(x) + \varepsilon, \quad 0 < c < p(y) + \varepsilon.$$

~~$x+y$~~ and $\boxed{x \in bK, \quad y \in cK,}$

$$\left(\frac{b}{b+c} K + \frac{c}{b+c} K\right) \subseteq K.$$

$$x + y \in bK + cK$$
$$\subseteq (b+c) K$$

$$\Rightarrow \quad p(x+y) \le b + c < p(x) + p(y) + 2\varepsilon,$$

$$\Rightarrow \quad p(x+y) < p(x) + p(y) + 2\varepsilon, \quad \forall \varepsilon > 0,$$

$$\Rightarrow \quad p(x+y) \le p(x) + p(y).$$

(3) If K is balanced,

then $ax \in cK \iff |a|x \in cK$.

for $x \in V$, $a \in \mathbb{K}$, $c > 0$

Thus, $p(ax) = p(|a|x) = |a|\, p(x)$.

$\Rightarrow$ p is a seminorm.

(2) Let $y \in K$ ~~of~~ (K consists of internal points)

$y + \varepsilon y \in K$ for sufficiently small ε.

$\Rightarrow y \in \frac{1}{1+\varepsilon} K \Rightarrow p(y) \leq \frac{1}{1+\varepsilon} < 1.$

$$\boxed{p(z) < 1}$$

$\Rightarrow \quad z \in cK$ for some $0 < c < 1$.

$\Rightarrow \quad \frac{1}{c} z \in K$ and $\frac{1}{c} > 1$.

$z \in [0, \frac{1}{c} z] \subseteq K$,

$\blacksquare$

(Hahn-Banach extension theorem for linear functionals)