03/02/2022

Lecture - 5

The Fourier transform

Let $f \in L^1(\mathbb{R})$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \boxed{e^{-2\pi i x\xi}}\, dx \quad \left[\int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, \xi\rangle}\, dx\right]$$

$$\hat{f} : \mathbb{R} \to \mathbb{C}.$$

$$\hat{f}(\xi + h_n) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi} \cdot e^{-2\pi i x h_n}\, dx \quad h_n \in \mathbb{R}.$$

$h_n \to 0$ as $n \to \infty$

$$\hat{f}(\xi + h_n) - \hat{f}(\xi) = \int_{\mathbb{R}} \underbrace{f(x)\, e^{-2\pi i x\xi}\left(e^{-2\pi i x h_n} - 1\right)}_{g_n}\, dx.$$

g_n, $|g_n(x)| \leq 2|f(x)|$, $x \in \mathbb{R}$.

$$\lim_{n\to\infty} \hat{f}(\xi+h_n) - \hat{f}(\xi)$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}} g_n(x)\,dx$$

$$= \int_{\mathbb{R}} \lim_{n\to\infty} g_n(x)\,dx = 0$$

[$\hat{f}$ is continuous at ξ.].

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}} f(x)\underline{\underline{e^{-2\pi i x\xi}}}\,\underline{\underline{dx}} \right|$$

$$\leq \int_{\mathbb{R}} |f(x)|\,dx = \|f\|_{1}, \ \forall \xi \in \mathbb{R}$$

$$\boxed{\|\hat{f}\|_\infty \le \|f\|_1}$$

$\mathcal{F} : L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$

$\mathcal{F}$ is a linear map.

$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$

$$\left[\begin{array}{l} \|f_n - f\|_1 \to 0 \\ \|\hat{f}_n - \hat{f}\|_\infty \to 0 \end{array}\right]$$

Prop^n: (Riemann–Lebesgue lemma) If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is a continuous function vanishing at infinity.

Proof: $f = \chi_{[a,b]}$ $-\infty < a < b < \infty$.

$$\hat{\chi}_{[a,b]}(\xi) = \int_a^b e^{-2\pi i x\xi}\,dx = \frac{1}{-2\pi i \xi}\left[e^{-2\pi i \xi b} - e^{-2\pi i \xi a}\right]$$

$$\left| \widehat{\chi_{[a,b]}}(\xi) \right| \le \frac{1}{\pi|\xi|}$$

$$\lim_{|\xi|\to\infty} \widehat{\chi_{[a,b]}}(\xi) = 0.$$

If f is a simple measurable function with compact support, then $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$

If $f \in L^1(\mathbb{R})$, for every $\varepsilon > 0$, $\exists$ a simple measurable fn. S_ε with compact support s.t. $\|f - S_\varepsilon\|_1 < \varepsilon.$ $\Rightarrow \|\hat{f} - \boxed{\widehat{S_\varepsilon}}\|_\infty \le \|f - S_\varepsilon\|_1 < \varepsilon.$

$$\boxed{\begin{array}{l}\text{Given } \varepsilon' > 0, \quad \exists\, N \text{ s.t.} \\ \qquad |\hat{f}(\xi)| \le \varepsilon', \quad \forall\, |\xi| \ge N.\end{array}}$$

$$|\hat{S}_\varepsilon(\xi)| \le \varepsilon/2 \quad \forall\, |\xi| \ge N.$$

$$\| \hat{f}(\xi) - \hat{S}_\varepsilon(\xi) \| \le \varepsilon/2$$

$$|\hat{f}(\xi)| \le |\hat{S}_\varepsilon(\xi)| + \varepsilon \quad \cancel{\le \varepsilon + \varepsilon'}, \quad \forall\, |\xi| \ge N.$$

$$< \varepsilon.$$

$\blacksquare$

$\mathcal{F}: L^1(\mathbb{R}) \to C_0(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$

$[\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})]$

$S(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$

$[\mathcal{F}: L^p(\mathbb{R}) \to L^q(\mathbb{R})]$

$1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$

$\mathcal{F}: L^3(\mathbb{R}) \hookrightarrow ??$ is not a function space

not a subset of $M(\mathbb{R})$

(application of closed graph theorem).

$\boxed{\frac{1}{\log(1+|x|)}}$

$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \ni \boxed{S(\mathbb{R})}$

$L^2([0,1]) \subseteq L^1([0,1])$

$\boxed{\ell^1(\mathbb{N})}$ $\ell^2(\mathbb{N})$

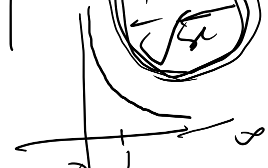
$1, \frac{1}{2}, \dots, \frac{1}{n}$

$a_1 + \dots + a_n < \infty$

$\to 0$

$a_n^2 < a_n$

$\frac{1}{\sqrt{x}}$



$$\ell^1(\{1,\dots,n\}) \overset{\sim}{=} \ell^2(\{1,\dots,n\})?$$

Central theme: Study of infinite system of linear equations.

$$\int_I \boxed{f(x)}\, g_\alpha(x)\, dx = c_\alpha. \qquad \rho(g_\alpha) = c_\alpha$$

where the $g_\alpha \in L^q(I)$, and one looks for a soln $f \in L^p(I)$. $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$\int_0^1 \boxed{x^n} \boxed{d\mu(x)} = c_n,$$

find $d\mu$ where $f \in L^p(I)$

$$x^n \longrightarrow \int_0^1 x^n \, d\mu(x)$$

$$g \longrightarrow \int_0^1 g \, d\mu(x)$$

ρ

$$\rho(x^n) = c_n$$

All functionals on $\boxed{C[0,1]}$ are of the form $g \longrightarrow \int_0^1 g \, d\mu.$

$p = q = 2$.

(1906) Hellinger–Toeplitz.

If a sequence (a_n) is such that the series $\sum_n a_n x_n$ is convergent for all sequences in $l^2(\mathbb{N})$, then $(a_n) \in l^2$.

Landau. More generally, if $\sum_n a_n x_n$ is convergent for all sequences (x_n) in $l^p(\mathbb{N})$, then

$$(a_n) \in l^q(\mathbb{N}) \ldots \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right), \quad 1 \leq p \leq \infty).$$

$(L^p(\mathbb{R}))^* \cong L^q(\mathbb{R})$.

$\int_0^1 e^{-2\pi i n x} d\mu(x) = \xi_n$

$\int_0^1 \boxed{x^n}\ \underline{d\mu}(x) = \boxed{c_n} \geq 0$.

$\exists\, M < \infty$, s.t.

$\rho(g_\alpha)$

$$\boxed{\left|\sum_{\alpha\in H} \lambda_\alpha g_\alpha\right| \leq M \sup_{x\in I} \left|\sum_{\alpha\in H} \lambda_\alpha \underline{g_\alpha}(x)\right|}$$

finite indexing set

$\in \operatorname{Span}\{1, x, \dots\}$

$\Leftarrow$ continuity?

$$\left|\rho\left(\sum_\alpha \lambda_\alpha g_\alpha\right)\right| \leq M \left\|\sum_{\alpha\in H} \lambda_\alpha g_\alpha\right\|_\infty$$

$\|\mathcal{F}(f)\|_\infty \leq \|f\|_1$

- Continuity of functionals. → on a function space
- Extension of functionals.

[Characterizing the space of functionals] $(C[0,1])^* \cong$ space of measures on $[0,1]$. - (Radon)

$L^p([0,1])^* \cong L^q([0,1])$

Abstract Functional Analysis

Vector space := over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$\mathbb{R}$ → real vector space

$\mathbb{C}$ → complex vector space

$V \to \mathbb{K}$-space

$X, Y \subseteq V$. For $a \in \mathbb{K}$, $aX := \{ax : x \in X\}$.

$X + Y := \{x + y : x \in X, y \in Y\}$.

$X - Y := X + (-1)Y$.

$x_1, \dots, x_n \in X$, $a_1, \dots, a_n \in \mathbb{K}$.

$a_1 x_1 + \dots + a_n x_n$ = (finite) linear combination of $x_1, \dots, x_n$ with coefficients $a_1, \dots, a_n \in \mathbb{K}$.

<u>Linear functional on V</u>:

$\rho : V \to \mathbb{K}$ linear map. $\big((\rho_1 + \rho_2)(x) = \rho_1(x) + \rho_2(x)\big)$

The set of all linear functionals on V is itself a vector space over $\mathbb{K}$, the <u>algebraic dual</u> space of V.

When ρ is non-zero then $Im(\rho) = \mathbb{K}$. $\Big\|$ $\rho(x) = \alpha \neq 0$
$\rho(\beta\alpha) = \beta\alpha$ $\alpha, \beta \in \mathbb{K}$

$\rho \longmapsto$ nullspace of $\rho \subseteq V$ subspace.

$\simeq$ (upto non-zero scalar ρ scaling) the above map is a one-to-one correspondence between linear functionals on V and codimension 1 subspaces of V,

<u>**Propn.**</u> If ρ is a linear functional on a $\mathbb{K}$-space V, then every linear functional on V that vanishes on the nullspace of ρ, is a scalar multiple of ρ. ~~Converse~~, each linear subspace of codimension 1 is the ~~subsp~~ nullspace of a nonzero functional on V.

Proof: Assume $\rho \neq 0$. $V_0 \to$ nullspace of ρ.

$V/V_0 \to \mathbb{K}$ $\qquad$ $V \to \mathbb{K}$ $\quad \rho_0(x+V_0) = \rho_0(y+V_0)$

$\cancel{\rho_0(x) = \rho_0(y)\ \rho(x-y)} = 0$

$\rho_0(x+V_0) = \rho(x)$.

$\rho_0 \to$ one-to-one linear operator from V/V_0 onto $\mathbb{K}$.

So, V/V_0 is one-dimensional. $\qquad$ ρ

$\rho_0 \to$ linear functional on V/V_0.

If σ vanishes on V_0, then there is a linear function σ_0 on V/V_0, $\sigma_0(x+V_0) = \sigma(x)$.

$\rho_1 : \mathbb{K} \to \mathbb{K} \qquad \rho_2 : \mathbb{K} \to \mathbb{K}$

$\rho_1(1) = \alpha \neq 0, \quad \underline{\rho_2(1) = \beta}$

$$\rho_2 = \frac{\beta}{\alpha} \rho_1$$

σ_0 is a scalar multiple of ρ_0.

$\rho_0 \quad V \to V/V_0$

$\sigma = \underline{\sigma_0 \circ Q} = a\rho_0 \circ Q = a\rho.$

V_1 codimension 1 subspace of V,

$V \xrightarrow{Q} V/V_1 \cong \mathbb{K}$

Let $\tau : V/V_1 \to \mathbb{K}$ be a nonzero map
$\tau \circ Q$ is a functional that vanishes on V_1.