01/02/2022

Lecture - 4

$f \in \mathcal{L}^p(X; \mu)$

$$\int_X |f|^p \, d\mu < \infty.$$

$l^p(A)$

$l^p(\mathbb{N}) \to$ complex sequence $\{\xi_n\}_{n \in \mathbb{N}}$.

$$\sum_{n=1}^{\infty} |\xi_n|^p < \infty.$$

$1 \leq p < \infty$

$\underline{L^\infty(X; \mu)}$ positive measure.

Suppose $g: X \to [0, \infty]$ is measurable.

$S = \{\alpha \in \mathbb{R} : \mu(g^{-1}(\alpha, \infty]) = 0\}$.

$\{x \in X : g(x) > \alpha\}$.

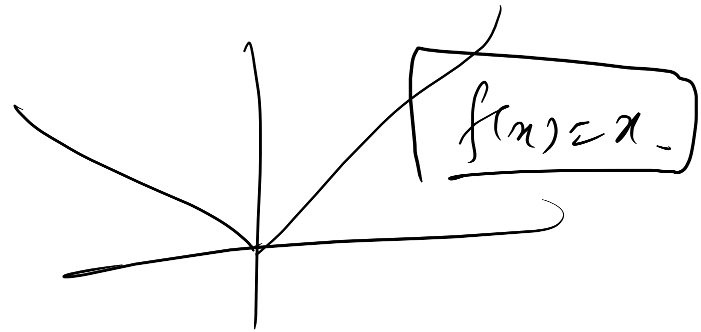
If $S = \phi$, define $\beta = \infty$. $\alpha \in S \Rightarrow \alpha' \geq \alpha$ is also in S.

If $S \neq \phi$, $\beta := \inf S$.

$\beta \to$ essential supremum of g.

If f is a complex measurable function on X,

$\|f\|_\infty :=$ essential supremum of $|f|$.



$L^\infty(\mu)$: $\{f$ complex measurable on X ; $\|f\|_\infty < \infty\}$.

"essentially" bounded measurable functions on X.

$|f(x)| \leq \lambda$ hold for almost all $x \in X$ if and only if $\lambda \geq \|f\|_\infty$..

<u>Theorem</u>: If p, q are conjugate exponents, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, $(p \geq \infty)$

$1 \leq p \leq \infty$, and $f \in L^p(\mu)$, $g \in L^q(\mu)$, then

$fg \in L^1(\mu)$, and

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

(Restatement of Hölder's inequality)

$\left(\|f\|_p = \| \, |f| \, \|_p\right)$

$f \in L^1(\mu)$, $g \in L^\infty(\mu)$

$$\int_X |f| \cdot |g| \, d\mu \leq \int |f| \cdot \|g\|_\infty \, d\mu = \|g\|_\infty \cdot \|f\|_1$$

Theorem: Suppose $1 \leq p \leq \infty$, $f, g \in L^p(\mu)$, then

$\Rightarrow$ $f+g \in L^p(\mu)$, and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p. \quad \left(\begin{array}{l} |f(x)+g(x)| \\ \leq |f(x)| + |g(x)| \end{array} \right)$$

$$\left(\int_X |f+g|^p \, d\mu \right)^{1/p} \leq \left(\int_X (|f|+|g|)^p \, d\mu \right)^{1/p}$$

$$\|\alpha f\|_p = \left(\int_X |\alpha|^p |f|^p \, d\mu \right)^{1/p} = |\alpha| \, \|f\|_p. \quad \leq \|f\|_p + \|g\|_p.$$

$L^p(\mu) \rightarrow$ complex vector space with a norm.

(normed linear space)

$(\|f\|_p = 0 \Leftrightarrow |f|^p = 0$ a.e. $\Leftrightarrow f = 0$ a.e.$)$

$\left[\underline{L^1(X;\mu)}, \boxed{L^2(X;\mu)}, \underline{L^\infty(X;\mu)} \right]$

$x^2+1=0$

Theorem: $L^p(X;\mu)$ is a complete metric space, for $1 \le p \le \infty$, (for every positive measure μ.

Proof: $\langle f_n \rangle$ — Cauchy sequence in $L^p(\mu)$, $(1 \le p < \infty)$.

(~~$\forall \varepsilon$~~ For every $\varepsilon > 0$, $\exists N$, s.t. $\forall m,n \ge N$ $\|f_m - f_n\|_p < \varepsilon$.)

$n = N$
$m = N+1$
$n_1 = N_{1/2}$

$\|f_{n_{i+1}} - f_{n_i}\|_p < 2^{-i}$ $\quad (n_{i+1} > n_i)$

$i = 1, 2, \ldots \ldots$

$g_k(x) = \sum_{i=1}^{k} |f_{n_{i+1}}(x) - f_{n_i}(x)|$, $\forall x \in X$

$\left[g(x) = \sum_{i=1}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)| \right]$ $x \in X$

$\|g_k\|_p < 1$ for $k = 1, 2, 3, \dots$

$$\int \liminf g_n \le \boxed{\liminf}$$

$\le \|g_k\|_p$

$\le 1.$

By Fatou's lemma;

$\|g\|_p \le 1 \Rightarrow g \in L^p(\mu)$

$\Rightarrow g(x) < \infty$ a.e.

$\Rightarrow f_{n_1}(x) + \sum_{i=1}^{\infty} f_{n_{i+1}}(x) - f_{n_i}(x)$

converges absolutely for almost every $x \in X$.

$\left[f(x) := \lim_{i \to \infty} f_{n_i}(x) \text{ a.e.} \right]$

To show, f is the L^p limit of $\langle f_n \rangle$.

$\epsilon > 0$, $\exists N$ s.t. $\|f_n - f_m\|_p < \epsilon$ $\Rightarrow$

for all $m, n \geq N$.

By Fatou's Lemma,

$$\int_X |f - f_m|^p \, d\mu \leq \liminf_{n \to \infty} \left(\int_X |f_n - f_m|^p \, d\mu \right)$$

$$\leq \epsilon^p$$

$\Rightarrow$ $\cancel{\|f - f_m\| \leq \epsilon}$.

$\|f - g\|_p \geq \left| \|f\|_p - \|g\|_p \right|$

$\|f\|_p \leq \|g\|_p + \|f - g\|_p$

$\Rightarrow \left| \|f\|_p - \|f_m\|_p \right| \leq \epsilon.$

$\Rightarrow f \in L^p(\mu)$.

$\|f - f_m\|_p \to 0$ as $m \to \infty$

$$\boxed{\lim_{m\to\infty} \|f - f_m\|_p} \leq \varepsilon \quad \forall \varepsilon > 0$$

$f_m \xrightarrow{\leq \text{ mon}} f$.

$\underline{p = \infty}$; $\boxed{A_k} = \{x \in X ; \underline{|f_k(x)| > \|f_k\|_\infty}\}$ ≤ uniformly bounded

$\boxed{B_{m,n}} = \{x \in X ; |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$

$E = (\bigcup A_n) \cup (\bigcup B_{m,n})$

$\mu(E) = 0$.

$|f_n(x) - f_m(x)| < \varepsilon$

$\forall m, n \geq N$.

On E^c, $\{f_n\}$ converges uniformly to a bounded function f. Define $f(x) = 0$ for $x \in E$.

Then $f \in L^\infty(\mu)$, and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

∎

Theorem: If $1 \le p \le \infty$ and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$ with limit f, then $\{f_n\}$ has a subsequence which converges pointwise a.e. to ~~...~~ f.

$L^p(\mu) \to \boxed{\text{Banach space}}$ $(1 \leq p \leq \infty)$

complete normed linear space.

Theorem :- $\int : S(X;\mu) \to \mathbb{C}$.

$L^1(X;\mu)$ is the completion of $S(X;\mu)$ in the L^1-norm.

$L^p(X;\mu)$ $(1 \leq p \leq \infty)$ is the completion of $S(X;\mu)$ in the L^p-norm.

Proof: $S \subset L^p(\mu)$, $s \in S$.

$1 \le p < \infty$, $|s|^p \in S$.

suppose $f \geq 0$, $f \in L^p(\mu)$.

$0 \le s_1 \le s_2 \cdots \le s_n \le \cdots f$

$s_n \to f$ pointwise.

$|f - s_n|^p \le |f|^p$ $\quad (|f|^p \in L^1(\mu))$

By DCT, $\lim_{n\to\infty} \left(\int_X |f-s_n|^p \, d\mu\right)^{1/p} = 0$

$\|f - s_n\|_p \to 0$ as $n \to \infty$.

$\boxed{f \in L^k \text{ closure of } S(X,\mu)}$

~~This~~ If f is complex measurable $(f = u + iv)$,

$$f \in L^k(\mu) \Leftrightarrow |f| \in L^k(\mu)$$

$$\Leftrightarrow \underbrace{u^+}_{=}, \underbrace{u^-}_{=}, \underbrace{v^+}_{=}, \underbrace{v^-}_{=} \in L^k(\mu)$$

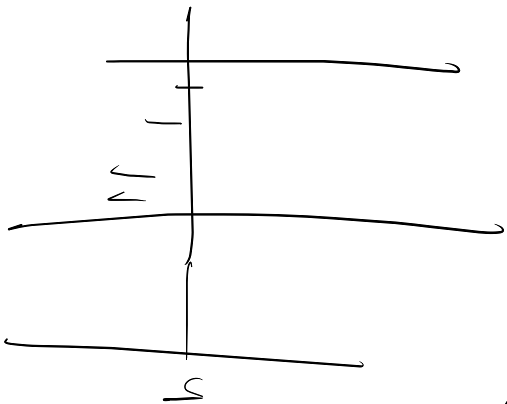
$$f = \underbrace{u^+}_{\substack{= \\ S_{u^+,n}}} - \underbrace{u^-}_{\substack{= \\ S_{u^-,n}}} + i(\underbrace{v^+}_{\substack{= \\ S_{v^+,n}}} - \underbrace{v^-}_{\substack{= \\ S_{v^-,n}}})$$

$f_n \to f$ in the L^k-norm

(f_n) simple measurable functions

then $f_n = S_{u^+,n} - S_{u^-,n} + i(S_{v^+,n} - S_{v^-,n})$

$p = \infty$



$|f(x)| \le M$ for almost every $x \in X$

$t \ge n$

$S_k = a_k \circ f$.

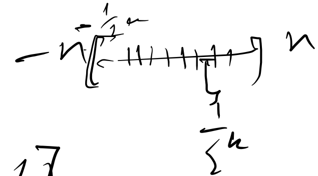
$\| S_n - f \|_\infty \le \frac{1}{2^k}$.

$\boxed{a_k(t) = \frac{[2^k t]}{2^k}}$

$|a_n(t) - t| < \frac{1}{2^n}$, $\forall t \in \mathbb{R}$

$|a_n \circ f(x) - f(x)| < \frac{1}{2^n}$ for almost all $x \in X$

$|a_n(t) - t| < \frac{1}{2^n}, \quad \forall t \in \mathbb{R},$



$$a_n(t) = \frac{[2^k t]}{2^n}.$$

$|f(x)| \le n$

$$\Big|\underbrace{a_k(f(x))}_{s_k} - f(x)\Big| < \frac{1}{2^n}, \quad \forall x \in X,$$

$a_n \circ f$ takes $2 \cdot 2^{k} n$ values.

Is s_n a simple measurable function?

Does s_n take finitely many values for almost all $x \in X$.

$$\boxed{\overline{S(X;\mu)}^{(\|\cdot\|_p)} = L^p(X;\mu)}$$

topology of local convergence in measure is coarser than the L^p-norm topologies.

$$L^p(X;\mu) \hookrightarrow \boxed{\overline{S(X)}}^{(\text{loc. conv.})}$$

space of μ-a.e. defined measurable functions on X.

Littlewood's three principles of real analysis

(1) Every (Lebesgue)-measurable set is "nearly" a finite union of intervals.
(inner, outer regularity)

(2) Every function (of class L^p) is "nearly" continuous. (Lusin's theorem)

(3) Every (pointwise) convergent sequence of functions is "nearly" uniformly convergent.
(Severini-Egorov theorem)

Continuity Properties of Measurable Functions

$X \to$ locally compact, Hausdorff space, $\mu \to$ +ve measure.

(1) (local finiteness)

$\mu(K) < \infty$ for every compact set $K \subset X$.

(2) (outer regularity)

For every $E \in \mathcal{M}$, we have

$$\mu(E) = \inf \{ \mu(V) ; E \subset V, V \text{ open} \}$$

(3) (inner regularity)

For every open set E, or for every $E \in \mathcal{M}$ with $\mu(E) < \infty$ we have

$$\mu(E) = \sup \{ \mu(K) ; K \subset E, K \text{ compact} \},$$

(4) (Completeness)

If $E \in \mathcal{M}$, $A \subset E$, $\mu(E) = 0$, then $A \in \mathcal{M}$.

$\mu \to$ complete Radon measure.

<u>Ex</u>: Lebesgue measure on $\mathbb{R}^n$.

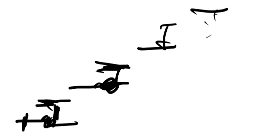
<u>Lusin's Theorem</u>: Suppose f is a complex measurable function on X, $\boxed{\mu(A) < \infty}$, $f(x) = 0$ outside of A, and $\varepsilon > 0$. Then there exists $g \in C_c(X)$ such that

$$\mu\left(\{x \in \overset{A}{\cancel{X}} : f(x) \neq g(x)\}\right) < \varepsilon.$$

(Furthermore, we may arrange g s.t. $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$.)

Proof: Assume $0 \le f < 1$, A is compact.

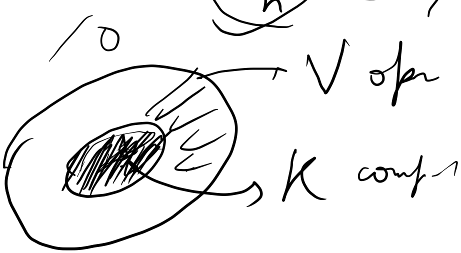
$$s_n = \varphi_n \circ f, \quad \varphi_n(t) = \begin{cases} \dfrac{[2^n t]}{2^n}, & 0 \le t < n \\ n, & n \le t \end{cases}$$



$t_1 := s_1, \dots, t_n := s_n - s_{n-1}$

$2^n t_n$ is a characteristic function of a set $T_n \subset A$.

[



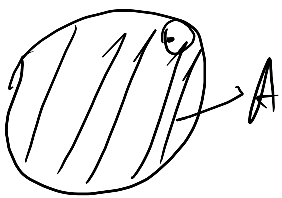
$\exists f \in C_c(X)$ s.t. $f(x) = 1$ on K, $f = 0$ on V^c, $0 \le f \le 1$.

]

$$f(x) = \sum_{n=1}^{\infty} t_n(x) \quad (x \in X).$$

↳ holds pointwise.

Let V be an open set, $A \subset V$, $\overline{V}$ is compact.



$\varepsilon > 0$. open ball of radius ε around x.

$x \in A$ $\quad B(x, \varepsilon)$

$$\bigcup_{x \in A} B(x, \varepsilon) \supseteq A, \qquad B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$$

Choose

$$\boxed{V = B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)}$$

$$\overline{V} \subseteq \overline{B(x_1, \varepsilon)} \cup \cdots \cup \overline{B(x_n, \varepsilon)}$$

$$A \subseteq \bigcup_{x_1, \dots, x_n} \overline{B(x_1, \varepsilon)}$$ is compact

There are compact sets K_n, ~~T_n~~ open sets V_n.

such that $K_n \subset T_n \subset V_n \subset V$ $(T_n \subset A \subset V)$

and $\boxed{\mu(V_n - K_n) < \frac{\varepsilon}{2^n}}$.

inner, outer regularity.

V_n

By Urysohn's lemma, there are functions $h_n \in C_c(X)$

such that $h_n \equiv 1$ on K_n, $h_n \equiv 0$ on V^c

$0 \le h_n \le 1$ on X.

$$g(x) := \sum_{n=1}^{\infty} 2^{-n} h_n(x) \qquad (x \in X)$$

The series converges uniformly on X, hence g is continuous and supported on $\overline{V}$ (which is compact.

In fact, $g \in C_c(X)$.

$\boxed{\text{Outside of } V_n - K_n,}$ $2^{-n} h_n(x) = \frac{1}{\sum n}$ on K_n.

$t_n(x) = \frac{1}{\sum n}$ on K_n

$\Rightarrow$ $\underline{\underline{2^{-n} h_n = t_n}}$ on ~~$V_n - K_n$~~ outside of $V_n - K_n$.

$$\sum_{n=1}^{\infty} 2^{-n} h_n = \sum_{n=1}^{\infty} t_n = \qquad \text{outside of } \left(\bigcup_{n=1}^{\infty} (V_n - K_n) \right)$$

$$\Rightarrow \quad g(x) = f(x) \qquad \text{outside of } \bigcup_{n=1}^{\infty} (V_n - K_n) \quad -$$

$$\underline{\underline{\mu\left(\bigcup_{n=1}^{\infty} (V_n - K_n) \right)}} \leq \sum_{n=1}^{\infty} \mu(V_n - K_n)$$

$$\leq \varepsilon$$

$\blacksquare$

Case II : If A is not compact but $\mu(A) < \infty$.

(f is bounded measurable fn. .)

Pf : K compact s.t. $\mu(A \setminus K) < \frac{\varepsilon}{2}$

$K \subset A$.

$\exists\ g \in C_c(X)$, s.t.

$$\mu(\{x \in K : f(x) \neq g(x)\}) < \frac{\varepsilon}{2}.$$

$$\mu(\{x \in A : f(x) \neq g(x)\}) < \varepsilon$$

Case II. ~~假~~ f is not bounded.

$B_n := \{x \in X ; |f(x)| > n\}$.

$$\bigcap_{n=1}^{\infty} B_n = \phi ,$$

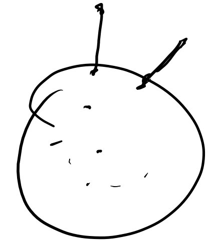
So, $\mu(B_n) \to 0$ as $n \to \infty$

f coincides with $\boxed{(1 - \chi_{B_n}) f}$ except on B_n,

$$\mu\{\{x \in A \cap B_n^c ; d(x) \neq g_n(x)\}\} < \frac{\varepsilon}{2}$$

$$\mu(B_n) < \frac{\varepsilon}{2} .$$

$$\boxed{R = \sup\{\,|f(x)| : x \in X\,\}}$$



$$a(z) = \begin{cases} z, & \text{if } |z| \le R \\ R\,\dfrac{z}{|z|}, & \text{if } |z| > R. \end{cases}$$

a is continuous on $\mathbb{C}$

$g_1 = a \circ g$ Whenever $g(x) = f(x)$ $\left(|g(x)| \le R\right)$

we have $g_1(x) = f(x)$.

& $|a(z)| \le R \ \forall z \in \mathbb{C}$,

$|g_1(x) = a \circ g(x)| \le R \quad \forall x \in X.$ ∎

Corollary. For $1 \le p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.

Proof. Given $s \in S$ and $\varepsilon > 0$, $\exists\, g \in C_c(X)$

s.t. $g_\varepsilon(x) = s(x)$ except on a set of measure $< \varepsilon$.

and $\boxed{|g_\varepsilon| \le \|s\|_\infty}$ $\forall x \in X$.

$|g(x) - s(x)| \le 2\|s\|_\infty$

$$\|g_\varepsilon - s\|_p = \left(\int_X |g(x) - s(x)|^p \, d\mu\right)^{1/p}$$

$$\le \left((2\|s\|_\infty)^p \varepsilon\right)^{1/p}$$

$$= 2\|s\|_\infty \cdot \varepsilon^{1/p}.$$

$\Rightarrow \quad \overline{C_c(X)}^{(\text{in } L^p\text{-norm})}$ contains $S(X)$.

$\Rightarrow \quad \overline{C_c(X)}^{(L^p\text{-norm})} = L^p(X, \mu)$.

$\blacksquare$

<u>Severini - Egorov theorem.</u>

Let $\{f_n\}$ be a sequence of complex measurable functions and suppose $A \subseteq X$ is a measurable subset with $\mu(A) < \infty$, and $f_n \to f$ μ-a.e. on A. Then for every $\varepsilon > 0$, $\exists$ a measurable subset B of A s.t. $\mu(B) < \varepsilon$, and $f_n \to f$ uniformly on $A \setminus B$.

Pf. Let $\varepsilon > 0$.

$$E_{n,k} := \bigcup_{m \geq n} \{x \in A : |f_m(x) - f_n(x)| \geq \tfrac{1}{k}\}.$$

$$|\cancel{f_m(x)}\underbrace{- f_n(x)| \geq \tfrac{1}{k}}$$

$$E_{n+1,k} \subseteq E_{n,k}$$

$\mu\left(\bigcap_{n \in \mathbb{N}} E_{n,k}\right) = 0$ because of pointwise conv. a.e.

for all $k \in \mathbb{N}$.

$\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$

$\left[B = \left(\bigcup_{k \in \mathbb{N}} E_{n_k,k}\right)\right]$, $\mu(B) < \varepsilon$.

$\boxed{x \notin B}$

$\Rightarrow x \notin E_{n_k,k}$ for any k.

$|f_m(x) - f_{n_k}(x)| < \frac{1}{k}$, $\forall\, m \geq n_k$. $\square$

topology of locally convergence in measure.

$\mu(F) < \infty$,

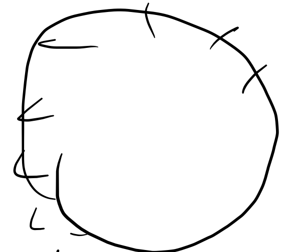
$N\{0; F, \varepsilon, \delta\}$

$= \{ f \text{ measurable};\ \mu(\boxed{\{|f(x)| > \varepsilon\} \cap F}) < \delta \}$,

In this topology,

the space of a.e.-defined measurable functions is a complete topological space. (and $S(X)$ is a dense subset.)

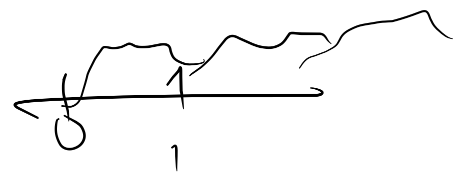
$\overline{S(X)}^{L^p} \hookrightarrow \overline{S(X)}^{\text{top loc. conv. on ...}}$



The Fourier transform:

(1) $f \in L^1(S^1, dx)$ (integrable periodic functions on $\mathbb{R}$)

$$\hat{f}(n) = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$$



$$\left[\hat{f} : \mathbb{Z} \to \mathbb{C}\right].$$

$\mathbb{R}^{\wedge}$

(2) Let $f \in L^1(\mathbb{R})$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx$$

$$\hat{f} : \mathbb{R} \to \mathbb{C}.$$

Does $\hat{f}$ make sense?

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx \right|$$

$$\leq \int_{\mathbb{R}} |f(x)| \, dx = \|f\|_1$$

$$\hat f(\xi+h) - \hat f(\xi)$$

$$= \int_{\mathbb R} \underbrace{f(x)\, e^{-2\pi i x \xi} \boxed{(e^{-2\pi i x h} - 1)}}_{g}\, dx$$

$$|g| \le 2|f(x)|$$

$$|\hat f(\xi+h) - \hat f(\xi)| \le 2\int_{\mathbb R} |f(x)|\, dx$$

DCT

$$\lim_{h\to 0} \hat f(\xi+h) - \hat f(\xi) = \int_{\mathbb R} \lim_{h\to 0} f(x)\, e^{-2\pi i x \xi}\,(e^{-2\pi i x h} - 1)\, dx = 0$$

$\Rightarrow$ $\hat f$ is continuous...