27/01/2022 Lecture-3

$$0 \leq s_1(x) \leq \cdots \leq s_n(x) \cdots \leq f(x)$$

$$\left(\lim_{n\to\infty} s_n(x) = f(x) \right)$$

Theorem (Lebesgue's DCT)

Suppose $\{f_n\}$ is a sequence of complex measurable function on X such that

$$\boxed{f(x) = \lim_{n\to\infty} f_n(x)}, \text{ exists for every } x \in X.$$

If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x), \quad (n \in \mathbb{N},\ x \in X), \text{ then}$$

$f \in L^1(\mu)$, $\left[\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0\right]$ and $\left[\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu\right]$.

Proof. $|f| \leq g \Rightarrow f$ is measurable.

$\Rightarrow f \in L^1(\mu)$. $\left(\int_X |f| \, d\mu < \infty \right)$

$$|f_n - f| \leq 2g.$$

Using Fatou's lemma for $\{2g - |f_n - f|\}$

$$\int_X 2g \, d\mu \leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|) \, d\mu$$

$$= \underline{\int_X 2g \, d\mu} - \left(\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \right)$$

$$\Rightarrow \left[\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0 \right]$$

$f : X \to \mathbb{C}$.

$$\Rightarrow \lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \qquad \left(\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu \right)$$

$$\leq \int_X |f_n - f| \, d\mu$$

$$\left[\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu \right]$$

証.

Null Sets (Sets of Measure Zero)

Let $(X, \mathcal{M}, \mu)$ be a measure space.

For $E \in \mathcal{M}$, if $\mu(E) = 0$, then E is said to be **μ-null**.

The collection of all $E \subset X$ for which $\exists$ $A, B \in \mathcal{M}$ s.t. $A \subset E \subset B$ and $\mu(B - A) = 0$ is denoted by $\mathcal{M}^*$. ($\mathcal{M} \subseteq \mathcal{M}^*$) ($E = A \cup (E \setminus A)$)

$E \setminus A \subseteq B \setminus A$.

Define $\mu^*(E) := \mu(A) = \mu(B)$.

Then $\mathcal{M}^*$ is a σ-algebra, and μ^* is a measure on $\mathcal{M}^*$ coinciding with μ on $\mathcal{M}$.

* The extended measure is said to be complete.
* All subsets of sets of measure 0 are now measurable.

Proof: (1) μ^* is well-defined for $E \in \mathcal{M}^*$.

$A \subset E \subset B$, $\quad A_1 \subset \boxed{E \subset B_1}$ $\quad (A, B, A_1, B_1 \in \mathcal{M})$

$\mu(B - A) = 0$, $\quad \boxed{\mu(B_1 - A_1) = 0}$.

$\boxed{A - A_1} \subset E - A_1 \subset B_1 - A_1$ $\quad 0 \leq \mu(A - A_1) \leq \mu(B_1 - A_1) = 0$

$\Rightarrow \mu(A - A_1) = 0 \Rightarrow \underline{\mu(A)} = \underline{\mu(A_1)} = \mu(A \cap A_1)$.

$\mathcal{M}^*$ is a σ-algebra

(a) $X \in \mathcal{M}^*$ $\quad (\mathcal{M} \subseteq \mathcal{M}^*)$.

(b) $E \in \mathcal{M}^*$ $\quad \exists A, B \in \mathcal{M}$ s.t. $A \subseteq E \subseteq B$

and $\mu(B - A) = 0$.

$B^c \subseteq E^c \subseteq A^c$ s.t. $\mu(A^c - B^c)$

$(= \mu(B - A) = 0)$

$\Rightarrow E^c \in \mathcal{M}^*$.

$(E_i \in \mathcal{M}^*)$

(c) $\forall A_i \subseteq E_i \subseteq B_i$, $(i \in \mathbb{N},\ A_i, B_i \in \mathcal{M})$

$A = \bigcup_{i=1}^{\infty} A_i$, $B = \bigcup_{i=1}^{\infty} B_i$, $E = \bigcup_{i=1}^{\infty} E_i$.

$A \subseteq E \subseteq B$

$B - A \subseteq \bigcup_{i=1}^{\infty} \underbrace{(B_i - A_i)}_{\mu\text{-null set}}$

$\therefore \mu(B - A) = 0.$

If E_i's are mutually disjoint, and $A_i \subseteq E_i \subseteq B_i$ so are the A_i's.

$\mu(E) = \mu(A) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu(E_i).$ $\blacksquare$

A.E. (Almost Everywhere)

If f, g are complex measurable functions on X

and if $\mu(\underbrace{\{x ; f(x) \neq g(x)\}}_{\mu\text{-null}}) = 0$, then we

say $f = g$, μ-a.e.

P: property which $x \in X$ may or may not have.

[P holds μ-a.e. if $\mu(\{x : x$ does not satisfy $P\}) = 0$]

Enlarging definition of measurability ; not necessarily complete

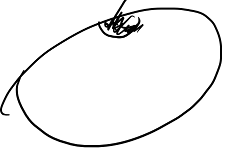
f defined on $E \in \mathcal{M}$ is said to be **measurable** on X if $\boxed{\mu(E^c) = 0}$ and if $f^{-1}(V) \cap E$ is measurable for every open set V.

$\cancel{X}$

$E^c \in \mathcal{M} = (E^c)^c = E \in \mathcal{M}$

$\mu : \mathcal{M} \to \mathbb{R}[0,\infty]$

$\mu(x) = 0$



$(f : X \to Y)$

X: meas. space

Y: top. space

$f^{-1}(V) \in \mathcal{M}$

$\forall$ open sets V in Y.

$f^{-1}(V) \in \mathcal{M}^*$

$\forall$ open sets V in Y

Theorem (DCT)

Suppose $\{f_n\}$ is a sequence of complex measurable functions $\boxed{\text{defined a.e}}$ on X such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Then the series

$$\left[f(x) = \sum_{n=1}^{\infty} f_n(x) \right]$$ converges for almost all $x \in X$.

$f \in L^1(\mu)$, and

$$\left[\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu \right].$$

Proof: f_n defined on $S_n \subseteq X$, $\mu(S_n^c) = 0$.

$$\varphi(x) := \boxed{\sum_{n=1}^{\infty} |f_n(x)|}, \quad \text{for } x \in S = \bigcap_{n=1}^{\infty} S_n$$

is defined a.e.

$$\mu(S^c) = \mu\left(\bigcup_{n=1}^{\infty} S_n^c\right) \le \sum_{n=1}^{\infty} \mu(S_n^c) = 0$$

(by MCT)

$$g_N(x) = \sum_{n=1}^{N} |f_n(x)|$$

$$g_1 \le g_2 \le \cdots \le g_N \le \varphi.$$

$$\int_S \varphi \, d\mu < \infty$$

$$= \int_X \varphi \, d\mu$$

$$\int \varphi \, d\mu \overset{MCT}{=} \lim_{N\to\infty}\left(\int g_N \, d\mu\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \int |f_n(x)| \, d\mu\right) < \infty$$

$E = \{x \in S : \varphi(x) < \infty\}$, $\mu(E^c) = 0$.

$\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for $x \in E$.

$$\boxed{|f(x)| \le c\, e(x)} \quad \text{on } E$$

$$\Rightarrow \quad f \in L^1(\mu) \ldots$$

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu .$$

[$S(X)$ = set of all complex measurable simple function on X.]

Theorem: $\|\cdot\|_1 : L^1(\mu) \to \mathbb{R}$ defined by
$$\|f\|_1 = \int_X |f|\, d\mu$$ has the following properties.

$\left(|(f+g)(x)| \le |f(x)| + |g(x)| \right)$

(1) $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

(2) $\|\alpha f\|_1 = |\alpha| \cdot \|f\|_1$

(3) $\|f\|_1 = 0 \iff \boxed{f = 0 \quad , \mu\text{-a.e.}}$

$L^1(\mu)$ $\ni g$

$d(f,g) = \|f-g\|_1$

$V \to$ complex vector space

$\|\cdot\| : V \to \mathbb{R}$ $\Big($ (1) $\|v+w\| \le \|v\| + \|w\|$ (2) $\|\alpha v\| = |\alpha| \cdot \|v\|$, (3) $\|v\| = 0 \iff v = 0$ $\Big)$

$L^1(\mu) \to$ normed linear space

$\hookrightarrow$ elements are equivalence classes of functions

μ-a.e. $\quad f, g \in L^1(\mu)$, $\quad f \sim g$

if $\quad \mu(\{x : f(x) \neq g(x)\}) = 0$

i.e., $\quad f = g \quad \mu$-a.e.

Topology of local convergence in measure

$S(X)$ — complex vector space $\qquad \text{range}(f) \subseteq \{\alpha_1, \dots, \alpha_m\}$

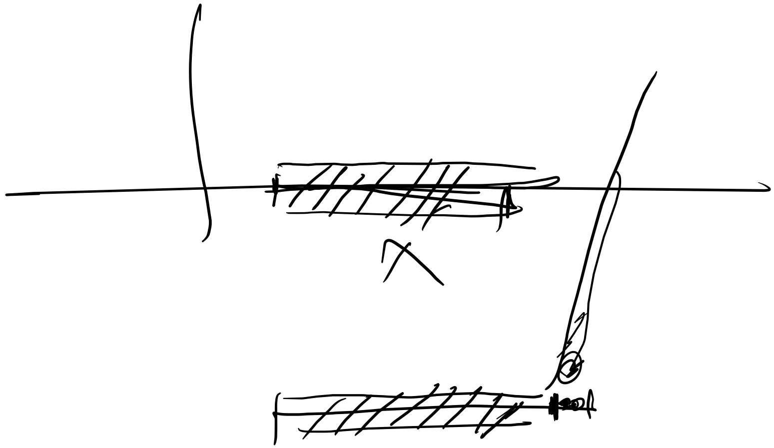
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{range}(g) \subseteq \{\beta_1, \dots, \beta_n\}$

$\mu(F) < \infty$

$$\{N(0; F, \varepsilon, \delta) = \{s : \mu(\{x : |s(x)| > \varepsilon, x \in F\}) < \delta\}$$

$\mathcal{B}$

S is "small" if on a big subset of X it is uniformly small.



$S(X) \to$ topological σ-algebra.

completion of $S(X)$ in this topology gives to a complete topological σ-algebra $\to$ (measurable functions on X), undefined on a ...

$\|\cdot\|_1$ defines a finer topology on $S(X)$ as compared to the topology of local convergence in measure.

$$\left[(S(X), \|\cdot\|_1) \right]^{-} \hookrightarrow \mathcal{M}(X)$$

$\mathcal{M}(X)$ defined measurable functions μ-a.e.

$\left[(S(X), \|\cdot\|_1) \right]^{-} \to L^1(\mu)$

Goal: Show that $\boxed{L^1(\mu)}$ is a complete metric space with $d(f,g) = \|f-g\|_1$.

L^p-spaces

$$\|\cdot\|_1 : L^1(\mu) \to \mathbb{R}$$

$$\|f\|_1 = \int_X |f| \, d\mu < \infty$$

Convex Functions and Inequalities:

Defn: A real function φ defined on the segment (a,b), where $-\infty \le a < b \le \infty$, is called <u>convex</u> if

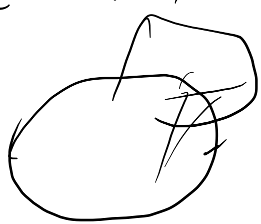
$$\varphi((1-\lambda)x + \lambda y) \le (1-\lambda)\varphi(x) + \lambda \varphi(y)$$

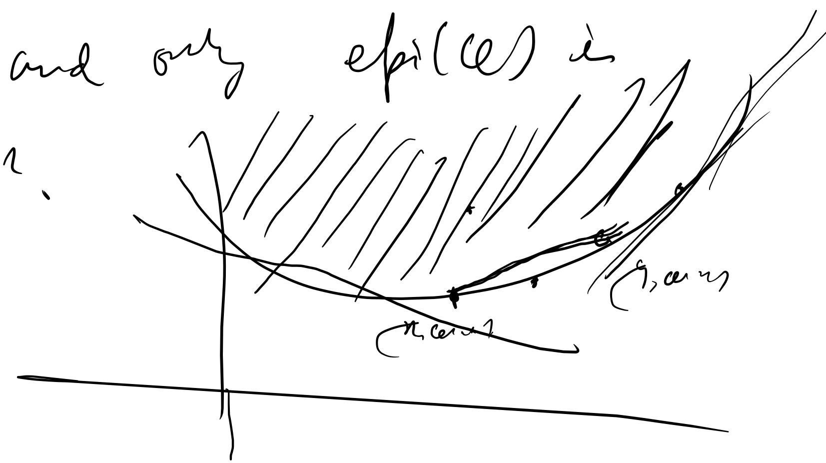
holds for $x, y \in (a,b)$, $\lambda \in [0,1]$.

For $f:(a,b)\to\mathbb{R}$, the epigraph of f,

$$\mathrm{epi}(f) = \{(x,y)\in\mathbb{R}^2 : y \ge f(x)\}.$$

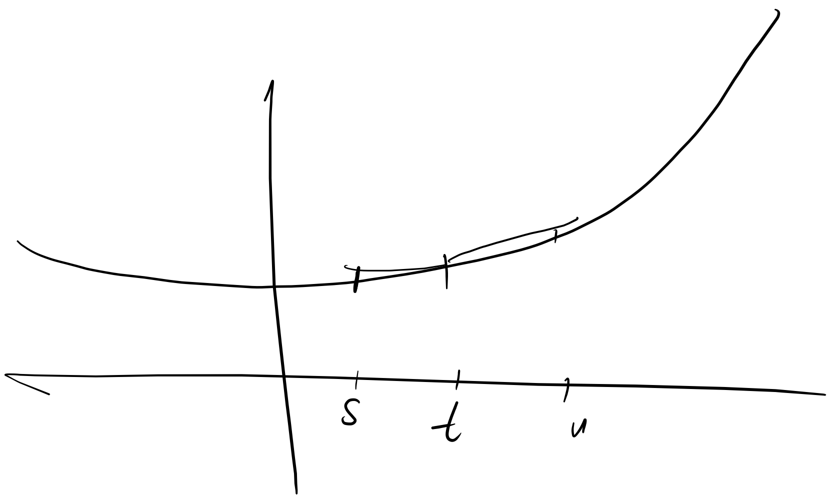
* φ is convex if and only if $\mathrm{epi}(\varphi)$ is a convex set in $\mathbb{R}^2$.

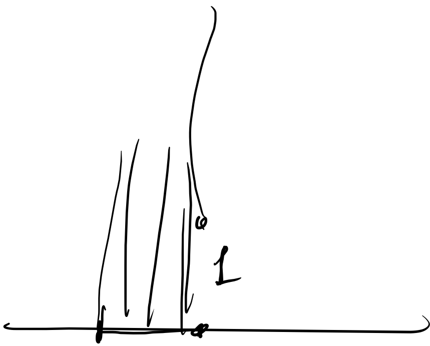




$$\left[\frac{\varphi(t)-\varphi(s)}{t-s} \leq \frac{\varphi(u)-\varphi(t)}{u-t} \right]$$

for $a < s < t < u < b$.





Theorem (Jensen's inequality)

Let μ be a positive measure on a σ-algebra $\mathcal{M}$ on Ω, so that $\mu(\Omega) = 1$. If f is a real-valued function in $L^1(\mu)$ with $a < f(x) < b$ $\forall x \in \Omega$, and if φ is convex on (a, b), then

$$\varphi\left(\int_\Omega f \, d\mu\right) \leq \int_\Omega (\varphi \circ f) \, d\mu$$

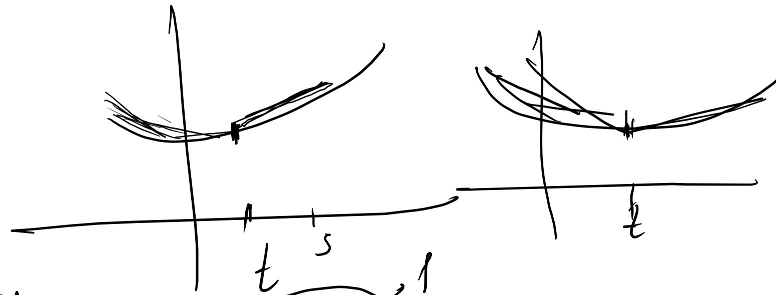
Proof: $\boxed{t = \int_\Omega f \, d\mu}$, $a\mathbb{1} < f < b\mathbb{1}$

$\Rightarrow a\,\mu(\Omega) < \underbrace{\int_\Omega f \, d\mu}_{t} < b\,\mu(\Omega)$

$\Rightarrow a < t < b.$

$$\varphi(s) \geq \varphi(t) + \beta(s-t) \qquad (a < s < b)$$

$$\varphi(f(x)) \geq \varphi(t) + \beta(f(x) - t)$$



$$\int_\Omega \varphi(t)\,d\mu = \varphi(t)\,\overset{1}{\mu(\Omega)}$$

$$\Rightarrow \int_\Omega (\varphi \circ f)(x)\,d\mu \geq \boxed{\varphi(t)} + \beta\left(\int_\Omega f(x)\,d\mu - t\right)$$

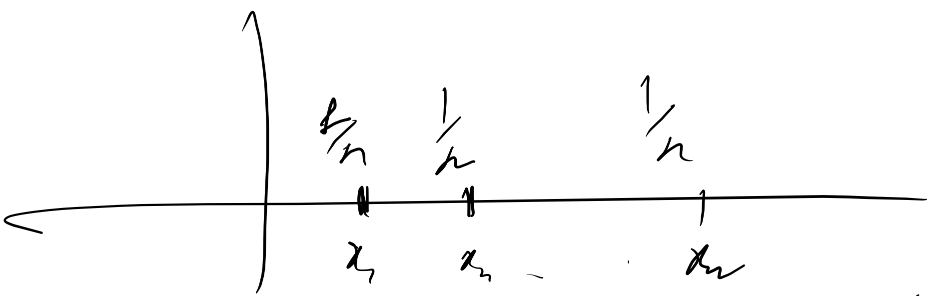
$$= \varphi\left(\int_\Omega f\,d\mu\right)$$

∎

If $\varphi = \exp$, then

$$\boxed{\exp\left(\int_\Omega f \, d\mu\right) \le \int_\Omega e^f \, d\mu}$$

μ — normalized counting measure on a finite subset of $\mathbb{R}$



$$\mu(\{x_i\}) = \frac{1}{n}$$

$$\mu(\{x_1, x_2\}) = \frac{2}{n}$$

$$f : \{x_1, \dots, x_n\} \to \mathbb{R}$$

$$f(x_i) = x_i.$$

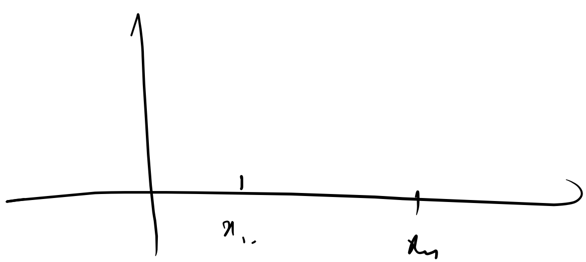
$$\exp\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{1}{n}\left(e^{x_1}+\cdots+e^{x_n}\right)$$

$$y_1 = e^{x_1}$$

$$\cdots\ y_n = e^{x_n}.$$

$$\left(y_1 \cdots y_n\right)^{1/n} \le \frac{y_1+\cdots+y_n}{n}$$

GM AM $y_i \ge 0$.



$u(\alpha x_1 \ldots$ $\alpha_i > 0$

$$\sum_{i=1}^{n} \alpha_i = 1.$$

$$\boxed{y_1^{\alpha_1}\cdots y_n^{\alpha_n} \le \alpha_1 y_1 + \cdots + \alpha_n y_n}$$

$y_i \ge 0$

$\alpha_1 + \cdots + \alpha_n = 1$

$\alpha_i \ge 0$,

Theorem: Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ (conjugate exponents). Let X be a measure space with positive measure μ. Let f, g be measurable function on X, with range in $[0, \infty]$. Then

$$\boxed{\int_X fg\,d\mu} \le \Big(\underbrace{\int_X |f|^p d\mu}_{A}\Big)^{1/p} \cdot \Big(\underbrace{\int_X |g|^q d\mu}_{B}\Big)^{1/q}.$$

(Hölder's inequality)

Proof: If $A = 0$, then $f = 0$ μ-a.e.

$\Rightarrow$ $fg = 0$ μ-a.e.

If $A > 0$, $B = \infty$, then also straightforward.

Assume $0 < A < \infty$, $0 < B < \infty$.

$$F := \frac{f}{A}, \quad G := \frac{g}{B}.$$

$$\int_X F^p \, d\mu = \int \frac{f^p}{A^p} \, d\mu = \frac{1}{A^p} \int f^p \, d\mu = \frac{A^p}{A^p} = 1.$$

$$\int_X G^q \, d\mu = 1$$

(Youngs inequality)

$\alpha\beta \le \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q$

$\alpha, \beta \ge 0,$

$\frac{1}{p}+\frac{1}{q}=1$

$$F(x) G(x) \le \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \quad \forall x \in X$$

$$\int_X FG \, d\mu \le \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = 1$$

$$\Rightarrow \int_X fg \, d\mu \le A \cdot B.$$

∎

Theorem (Minkowski's inequality) $1 < p < \infty$.

$$\left(\int_X (f+g)^p \, d\mu\right)^{1/p} \le \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}$$

Proof: $(f+g)^p = \boxed{f \cdot (f+g)^{p-1}} + g \cdot (f+g)^{p-1}$

By Hölder inequality

$$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$$

$$\int_X f \cdot (f+g)^{p-1} \, d\mu \le \left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X (f+g)^{(p-1)q} \, d\mu\right)^{1/q} = \left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X (f+g)^p \, d\mu\right)^{1/q}$$

$$\int_X g \cdot (f+g)^{p-1} \, d\mu \le \left(\int_X g^p \, d\mu\right)^{1/p} \left(\int_X (f+g)^p \, d\mu\right)^{1/q}.$$

$$\left(\int_X (f+g)^p \, d\mu\right)^{1}$$

$$\le \left(\left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}\right)\left(\int_X (f+g)^p \, d\mu\right)^{1/q},$$

$$\left(\frac{f+g}{2}\right)^p \le \frac{f^p + g^p}{2},$$

t^p is convex on $(0,\infty)$ for $p \ge 1$.

If $\int_X f^p \, d\mu < \infty$, $\int_X g^p \, d\mu < \infty$,

then $\int_X (f+g)^p \, d\mu < \infty$.

L^p-spaces

Def. If $1 < p < \infty$ and f is a complex measurable function on X we define

$$\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}$$

and let $L^p(\mu)$ consists of all f for which

$$\|f\|_p < \infty.$$

L^p-norm of f.

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$$\|\alpha f\|_p = |\alpha| \cdot \|f\|_p.$$

$\|f\|_p = 0 \iff f = 0$ μ-a.e.,

Remarks :

(1) If μ is the Lebesgue measure on $\mathbb{R}^n$, $[\rho : C_c(\mathbb{R}^n) \to \mathbb{C}]$

then $L^p(\mathbb{R}^n)$ is used instead

of $L^p(\mu)$.

(2) If μ is the counting measure on a set

A, $L^p(A)$ is denoted by $\ell^p(A)$.

$A = \mathbb{Z}$ or $\mathbb{N}$) $\ell^p(\mathbb{Z})$, $\ell^p(\mathbb{N}) = \ell^p$.

$\ell^p(\mathbb{N}) \to$ complex sequence $\{\xi_n\}$ s.t.

$$\sum_{n=1}^{\infty} |\xi_n|^p < \infty.$$

$$\|(\xi, \ldots)\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p}.$$

Function spaces:

(1) $C_c(\mathbb{R}^n)$ — continuous fn. with compact support.

(sup norm)

(2) $\cancel{L^1(X,\mu)}$ — $\|\cdot\|_1$

(3) $L^p(X,\mu)$ — $\|\cdot\|_p$

(4) $\ell^p(\mathbb{Z})$, $\ell^p(\mathbb{N})$

Moment problem.

$E(X^n) = c_n$

$f \quad C([0,1])$

$\int_0^1 \boxed{x^n}\, d\mu = c_n$

$\rho(x^n) = c_n$

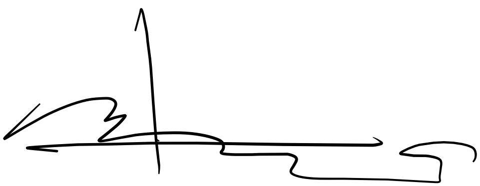
$\int \phi(x)\, d\mu$

$$(\mu, \mathcal{M}) \xrightarrow{\text{complete}} (\mu^*, \mathcal{M}^*) \qquad E \in \mathcal{M}^* \setminus \mathcal{M}$$

$$\chi_E$$

$$\int_X \left(\sum_{i=1}^{n} a_i \chi_{A_i} \right) = \sum_{i=1}^{\infty} a_i \, \mu(A_i) \qquad \cancel{+\infty - \infty}$$

$$\int \chi_E = \mu(E)$$



$$\cancel{\mu(E) = \varphi(E) + \varphi^{-1}(E)}$$