25/01/2022

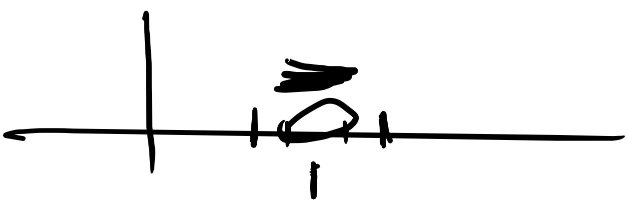
Lecture - 2

$\left[\underline{\underline{u}} : \Omega \longrightarrow F \right]$

$\subseteq E$

$\int : \underline{\underline{L^1(X; \mu)}} \longrightarrow \mathbb{C}$

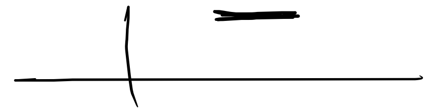
integration is a linear functional..



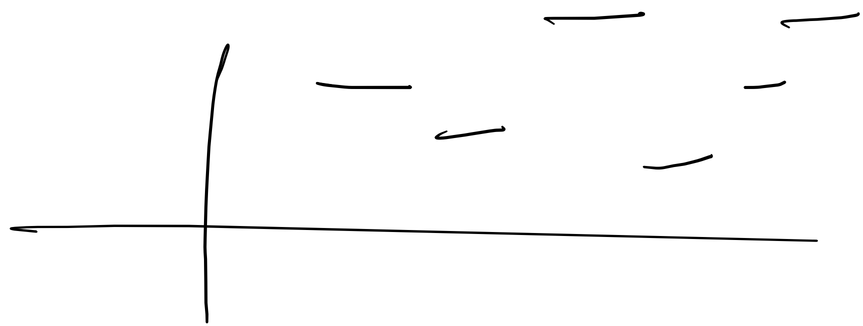
$K = supp(f)$

$\int \chi_K = m(K)$

(1) We have a clear idea how to integrate indicator functions.



(2) We can extend the definition of integration to finite linear combinations of indicator function (simple function or step function)



$X \longrightarrow$ locally compact Hausdorff space

$C_c(X) \longrightarrow$ complex-valued continuous functions on X with compact support.

Riesz <u>representation theorem</u>:

Given a Radon measure μ on X,

$$\int \cdot \, d\mu : C_c(X) \longrightarrow \mathbb{C}.$$

positive linear functional's

Conversely, given a positive linear functional on $C_c(X)$, there corresponds a Radon measure.

A Crash Course in the Theory of Integration

The Concept of Measurability.

(a) $X \to$ set

A collection $\mathcal{M}$ of subsets of X is said to be a σ-algebra on X if $\mathcal{M}$ has the following properties.

(i) $X \in \mathcal{M}$

(ii) if $A \in \mathcal{M}$, then $X \setminus A \in \mathcal{M}$.

(iii) if $A = \bigcup_{n=1}^{\infty} A_n$, for $A_n \in \mathcal{M}$, then $A \in \mathcal{M}$.

(b) If $\mathfrak{M}$ is a σ-algebra in X, then X is called a *measurable space*, and the members of $\mathfrak{M}$ are called the *measurable sets* in X.

(c) If X, Y are measurable spaces, and $f : X \to Y$ is a mapping. f is said to be *measurable* if $f^{-1}(U)$ is a measurable set in X for every measurable set U in Y.

<u>Theorem</u>: Let Y, Z be $\boxed{\text{topological}}$ (measurable) spaces and $g : (Y, \mathcal{B}_Y) \to (Z, \mathcal{B}_Z)$ be $\boxed{\text{continuous}}$ (meas). If X is a measurable space, and $f : X \to Y$ is measurable and $h = g \circ f$, then $h : X \to Z$ is measurable.

If Y is a topological space, then the σ-algebra generated by (τ) open sets is called the <u>Borel ~~set~~ σ-algebra</u> of Y. $(\mathcal{B}_Y)$.

$\ell^\infty(\mathbb{N}) \cong C(\beta\mathbb{N})$ | $\mathcal{B}(X) \to$ bounded measurable complex valued

$\to$ ($*$-algebra) $(\cong C(Y))$ $\to$ compact Hausdorff space.)

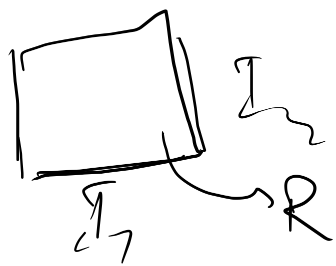
Theorem: Let u, v be real measurable functions on a measurable space X, let Φ be a cont. mapping of $\mathbb{R}^2$ onto a top. space Y, and define $h(x) = \Phi\left(\boxed{u(x), v(x)}\right)$, for $x \in X$.

Then $h: X \to Y$.

Pf: –

$f(x) = (u(x), v(x))$

$X \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\Phi} Y$



$f^{-1}(R) = \boxed{u^{-1}(I_1)} \cap v^{-1}(I_2)$

Remarks: (Building new measurable functions from old)

(a) If $f = u + iv$, where u, v are real measurable functions, then f is a complex measurable function on X.

(b) If $f = u + iv$ is a complex measurable function, then $u, v, |f|$ are ~~complex~~ real measurable functions on X.

$r : \mathbb{C} \to \mathbb{R}$

$z \to \mathrm{Re}\, z$ $\quad z \mapsto \mathrm{Im}\, z$ $\quad z \to |z|$

(c) If f, g are complex measurable functions on X, then so are $f+g$, fg.

(d) If E is a measurable in X, and

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable function.

(e) If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$,

$$\boxed{f = \alpha|f|}$$

$\left|\int_X f\,d\mu\right| \le \int_X |f| \ldots$

Proof of e) $E = \{x ; f(x) = 0\}$ is measurable.

$$\varphi(z) = \frac{z}{|z|} \quad \text{on } \mathbb{C} \setminus \{0\}.$$

$$\alpha(x) = \varphi\left(f(x) + \chi_E(x) \right) \quad (x \in X)$$

Theorem: Suppose $\mathcal{M}$ is a σ-algebra in X, $Y = [-\infty, \infty]$. $f : X \to Y$ is a mapping. If $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for ~~a~~ every $\alpha \in \mathbb{R}$, then f is measurable.

Proof: $[-\infty, ~~\alpha~~ \beta) = \bigcap (\alpha_n, \infty] ; (\alpha, \beta)$.

Suppose $\{f_n\}$ is a sequence of extended-real functions on X. Then

$$\left(\sup_n f_n\right)(x) = \sup_n f_n(x)$$

$$\left(\limsup_{n\to\infty} f_n\right)(x) = \limsup_{n\to\infty} f_n(x)$$

If $\quad f(x) = \boxed{\lim_{n\to\infty} f_n(x)} \quad \forall x \in X$

then f is said to be the pointwise limit of f_n.

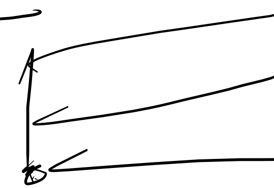
Theorem: If $f_n : X \to [-\infty, \infty]$ is measurable, for $n \in \mathbb{N}$, and $g = \sup_{n \geq 1} f_n$,

$h = \limsup_{n \to \infty} f_n$, then g and h are measurable.

Proof: $g^{-1}((\alpha, \infty]) = \underline{\bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])}$

$\in \mathcal{M}$.

f_1, f_n

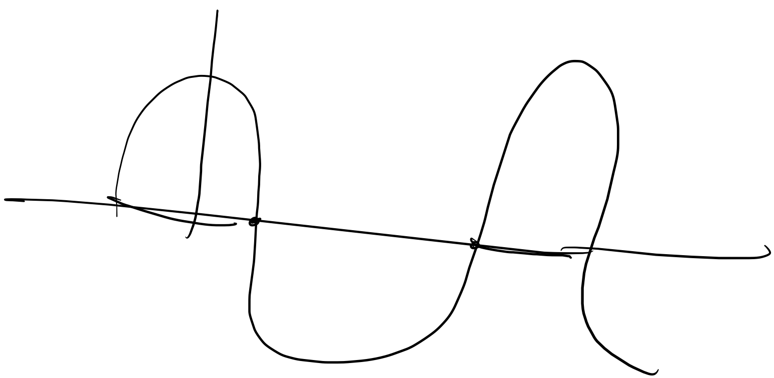


$$h = \inf_{k \geq 1} \left(\boxed{\sup_{i \geq k} f_i} \right)$$

Corollary

(1) Limit of every pointwise convergent sequence of complex measurable functions is measurable.

(2) If f, g are ~~complex~~ real measurable, then $\max\{f, g\}$ is also real measurable.



$f^+ = \max\{f, 0\}$ } are real measurable.

$f^- = -\min\{f, 0\}$

$f = f^+ - f^- = g - h$

$g, h \geq 0$,

$g \geq f^+, h \geq f^-$.

SIMPLE FUNCTION:

Def^n: A complex measurable function s on X whose range consists of only finitely many points (not including $-\infty, \infty$) is called a simple function.

If $\alpha_1, \ldots, \alpha_n$ are the distinct values of s.

$$A_i = \{x : s(x) = \alpha_i\} = s^{-1}(\{\alpha_i\})$$

$$\boxed{s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}}$$

Theorem: Let $f: X \to [0, \infty]$ be measurable. Then there exist measurable functions s_n on X such that

(1) $0 \le s_1 \le s_2 \le \cdots \le f$

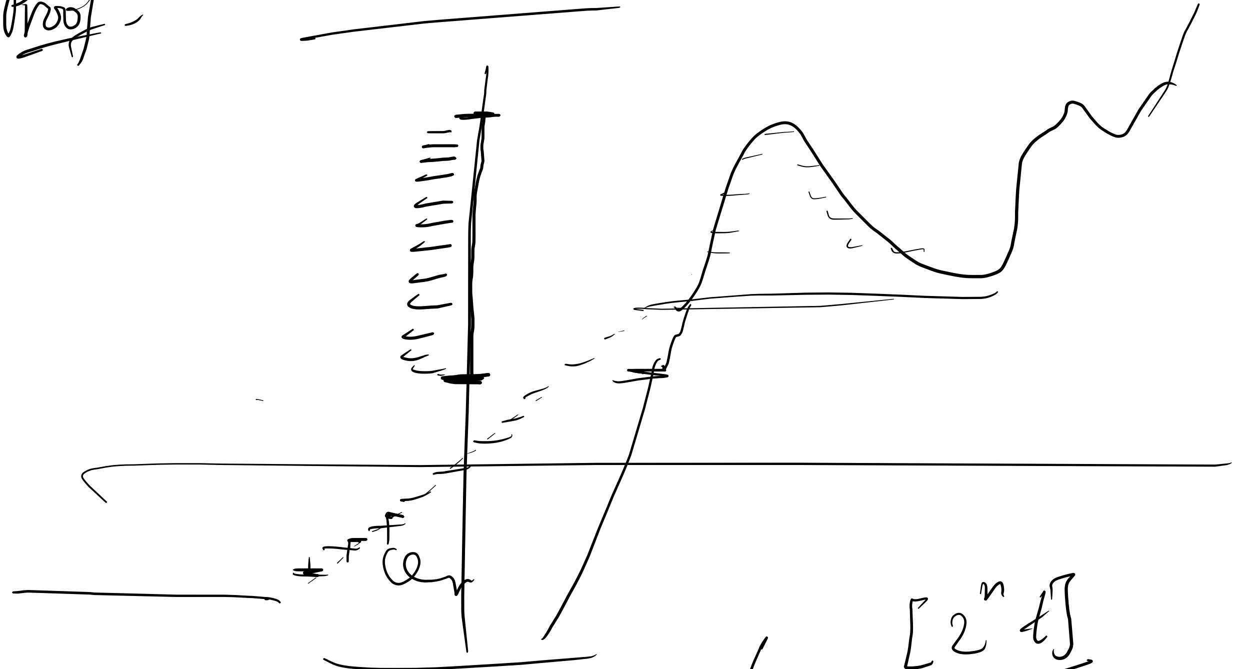
(2) $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

[$\mathbb{R} \to$ completion of $\mathbb{Q}$.

order completion (Dedekind cuts)

(metric completion (Cauchy sequences))]

Proof -



$$a_n(t) = \begin{cases} \dfrac{[2^n t]}{2^n}, & 0 \le t < n \\ n, & n \le t < \infty. \end{cases}$$

$\varphi_n \longrightarrow$ Borel function on $[0,\infty]$.

$$t - \frac{1}{2^n} \le \varphi_n(t) \le t$$

$$\left[\, 0 \le \varphi_1 \le \varphi_2 \le \cdots \le t \,\right].$$

and $\varphi_n(t) \to t$ as $n\to\infty$ $\forall t \in [0,\infty]$.

$$\boxed{S_n = \varphi_n \circ f}$$

$\varphi_n \circ f \longrightarrow$ idot pointw

$S_n \to f$

$$0 \le S_1 \le S_2 \cdots \le f.$$

Elementary Properties of Measures.

Defn

(a) A *positive measure* on X is a function $\mu : \mathfrak{M} \to [0, \infty]$ which is countably *additive* and not identically ∞.

$$\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset)$$

(b) A *measure space* is a measurable space which has a positive measure defined on its σ-algebra.

(c) A complex measure is a complex valued countably additive function on a σ-algebra.

Theorem. Let μ be a positive measure on a σ-algebra $\mathcal{M}$. Then

(a) $\mu(\phi) = 0$.

(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if $A_1, \ldots, A_n$ are pairwise disjoint.

(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$, if $A, B \in \mathcal{M}$.

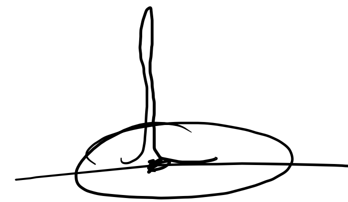
(d) $\mu(A_n) \longrightarrow \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{M}$, and $A_1 \subset A_2 \subset \cdots$.

(e) $A_1 \supset A_2 \supset \cdots$, $A_n \in \mathcal{M}$,

$\mu(A_1) < \infty$,

then $\mu(A_n) \to \mu(A)$ as $n \to \infty$

$\left(A = \bigcap_{n=1}^{\infty} A_n\right)$,



Examples:

(1) Counting measure.

(2) $x_0 \in X$, $\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$

Dirac measure at x_0.

(unit mass concentrated at x_0).

Arithmetic in $[0,\infty]$.

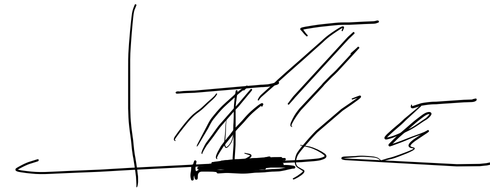
$$a+\infty = \infty + a = \infty, \quad \text{if } 0 \leq a \leq \infty$$

$$a \cdot \infty = \infty \cdot a = \begin{cases} \infty & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases}$$

commutativity, associativity, distributivity hold for $[0,\infty]$.

* All measurable functions are integration ready.

INTEGRATION OF POSITIVE FUNCTIONS



Defn: If $s : X \to [0,\infty)$ is a simple measurable function, $s = \sum_{i=1}^{n} a_i \chi_{A_i}$, and if $E \in \mathcal{M}$, we define

$$\boxed{\int_E s \, d\mu := \sum_{i=1}^{n} a_i \, \mu(A_i \cap E)}$$

If $f : X \to [0,\infty]$ is measurable, and $E \in \mathcal{M}$, we define

$$\int_E f \, d\mu = \sup \int_E s \, d\mu$$

supremum is taken over all simple measurable functions, s s.t. $0 \le s \le f$.

Proposition:

(a) If $0 \le f \le g$, then $\int_E f \, d\mu \le \int_E g \, d\mu$

(b) If $A \subseteq B$ and $f \ge 0$, then $\int_A f \, d\mu \le \int_B f \, d\mu$.

(c) If $f \ge 0$ and c is a constant, $0 \le c < \infty$, then $\int_E (c f) d\mu = c \int_E f \, d\mu$.

(d) If $f(x) = 0 \ \forall x \in E$, then

$$\int_E f \, d\mu = 0 \quad (\text{even if } \mu(E) = \infty)$$

(e) If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$

(even if $f(x) = \infty \ \forall x \in E$).

(f) If $f \geq 0$, then $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$.

Lebesgue's Monotone Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable function on X, and suppose that

(a) $0 \le f_1(x) \le f_2(x) \le \dots \le \infty$ for every $x \in X$,

(b) $f_n(x) \longrightarrow f(x)$ as $n \to \infty$, for every $x \in X$.

Then f is measurable, and

$$\boxed{\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu} \text{ as } n \to \infty$$

$$\left[\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) d\mu \right]$$

Proof. $\int f_n \, d\mu \leq \int f_{n+1} \, d\mu, \quad \forall n \in \mathbb{N}.$

$\Rightarrow \exists \, \alpha \in [0, \infty]$ s.t. $\int_X f_n \, d\mu \to \alpha$

as $n \to \infty$.

T. Show. $\alpha = \int_X (\lim_{n\to\infty} f_n) \, d\mu = \int_X f \, d\mu$

f is measurable.

$$\boxed{\alpha \leq \int_X f \, d\mu}$$

$\int f_n \leq \int_X f.$

Let s be a simple measurable fn s.t.

$\boxed{0 \le s \le f}$. $\quad 0 < c < 1$.

$E_n = \{x : f_n(x) \ge c\,s(x)\}$

$\varphi(X) = \int_X s\,d\mu$

$\varphi(E) = \int_E s\,d\mu$

$E_1 \subset E_2 \subset \cdots$, $\quad \left[X = \bigcup_{n=1}^{\infty} E_n\right]$.

$$\int_X f_n\,d\mu \ge \int_{E_n} f_n\,d\mu \ge c \boxed{\int_{E_n} s\,d\mu}$$

$$\Rightarrow \quad \alpha \ge c \boxed{\int_X s\,d\mu},$$

$$\Rightarrow \quad \alpha \ge \int_X s\,d\mu. \quad \Rightarrow \alpha \ge \sup \int_X s\,d\mu = \int_X f\,d\mu$$

__Theorem:__ If $f_n : X \to [0,\infty]$ is measurable, for $n \in \mathbb{N}$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in X),$$ then

$$\boxed{\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu}$$

$$\sum_{n=1}^{\infty} f_n$$

Pf: $g_N = f_1 + \dots + f_N$ (Use MCT for g_N)

Fatou's lemma :

If $f_n : X \to [0, \infty]$ is measurable, for $n \in \mathbb{N}$,

then
$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

Pf :- $\quad g_k(x) = \inf_{i \ge k} f_i(x).$

$$\sup_{k \ge 1} g_k = \liminf_{n\to\infty} f_n$$

$$0 \le g_1 \le g_2 \le \cdots$$

$$\int_X \left(\sup_{k\ge 1} g_k\right) d\mu$$

$$\boxed{g_k \le f_k}$$

$$\le \boxed{\lim_{k\to\infty} \int g_k \, d\mu} \le \liminf_{k\to\infty} \int f_k \, d\mu$$

INTEGRATION OF COMPLEX FUNCTIONS

<u>Def</u>: $L^1(X,\mu) \to$ collection of all complex measurable functions f on X which satisfy,

$$\boxed{\int_X |f|\, d\mu < \infty}$$

members of $L^1(\mu)$ are called Lebesgue μ-integrable function or μ-summable functions.

Defn. If $f = u + iv$, and $f \in L^1(\mu)$,

$$\int_E f \, d\mu := \left(\int_E u^+ \, d\mu - \int_E u^- \, d\mu\right)$$

$$+ i \left(\int_E v^+ \, d\mu - \int_E v^- \, d\mu\right)$$

Theorem: Suppose $f, g \in L^1(X, \mu)$ $\alpha, \beta \in \mathbb{C}$. Then

$\alpha f + \beta g \in L^1(\mu)$, and

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \left(\int_X f \, d\mu\right) + \beta \left(\int_X g \, d\mu\right)$$

Theorem (Triangle inequality).

$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu$$

Proof.

$\alpha =$ $\left(\int_X f\,d\mu\right)$

$\left(\int_X f\,d\mu\right) = r e^{i\theta}$

$\boxed{\alpha = e^{-i\theta}}$

$$\left|\int_X f\,d\mu\right| = \alpha \int_X f\,d\mu \in \mathbb{R}$$

$$= \int_X \alpha f\,d\mu \in \mathbb{R},$$

$\alpha f = u + i v$

$$\boxed{\int_X u\,d\mu} + i \int_X v\,d\mu \in \mathbb{R}$$

$u = \mathrm{Re}(\alpha f) \le |\alpha f|$

$$\le \int_X |\alpha f|\,d\mu$$

Theorem (Lebesgue DCT).

Suppose $\{f_n\}$ is a sequence of complex measurable functions on X s.t.

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in X.$$

If there is a function $g \in L^1(\mu)$ s.t.

$$|f_n(x)| \le g(x), \quad (n \in \mathbb{N}),\ x \in X,$$

then $f \in L^1(\mu)$, and

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$$

$$\left(\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X \overbrace{\lim_{n\to\infty} f_n}^{f}\, d\mu \right.$$