18/01/2022

Lecture-1

Assignment - 30%

Mid-term - 20%

Final - 50%

What is functional analysis?

The study of topological vector spaces and
of mappings $u : \Omega \to F$ from a part Ω
of a topological vector space E onto a topological
vector space F, these mappings being assumed

to satisfy various algebraic and topological condition.

(def^n due to J Dieudonné)

The 1-D heat eq^n.

Metal rod with non-uniform temperature.

heat (thermal energy) is transferred from regions of higher temperature to lower temperature.

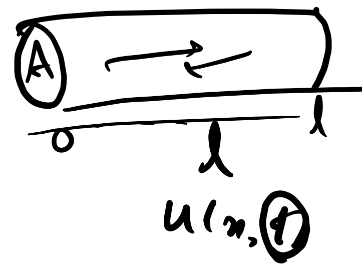
Three physical principles.

(1) Heat of a body = $c\,m\,u$

$c \to$ specific heat

$m \to$ mass of body

$u \to$ temperature



(2) Fourier's law of heat transfer:

rate of heat transfer is proportional to the negative temperature gradient.

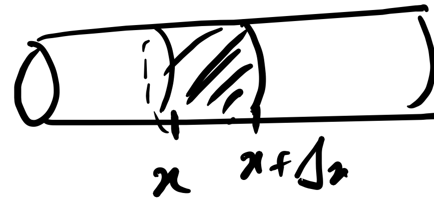
$$\text{rate of heat transfer}/\text{area} = -K_0 \frac{\partial u}{\partial x}$$

(3) Conservation of energy.

(i) uniform rod of length l

(ii) sides are insulated and ends are exposed.

(iii) no heat source.



heat energy of segment

$$= c \times (\rho A \Delta x) \times u(x,t)$$

By conservation of energy,

change of heat energy of segment in Δt,

= heat in from left boundary

— heat out from right boundary

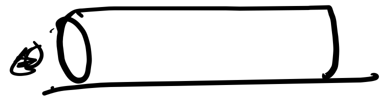
$$c \rho A \Delta x \, u(x, t+\Delta t) - c \rho A \Delta x \, u(x,t)$$

$$= (\Delta t) A \left(-K_0 \frac{\partial u}{\partial x}\right)_x - (\Delta t) A \left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$\Rightarrow \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \frac{K_0}{c\rho} \, \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}$$

$\Rightarrow$ for $\Delta x, \Delta t \to 0$, $\boxed{\dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}}$

Solving the heat eqn:



$\boxed{\text{IC}}$ $u(x,0) = f(x)$

$\boxed{\text{BC}}$ $u(0,t) = u(1,t) = 0$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$$

Separation of variables : $u(x,t) = X(x) \cdot T(t)$

$$\frac{\partial^2 u}{\partial x^2} = X''(x) \cdot T(t)$$

$$\frac{\partial u}{\partial t} = X(x) \cdot T'(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$$

$$X_n(x) = b_n \sin(n\pi x)$$

$$T_n(t) = c_n e^{-n^2\pi^2 t}$$

$$\left[u_n(x,t) = B_n \sin(n\pi x) e^{-n^2\pi^2 t} \right]$$

$$\boxed{u_n(x,0) = B_n \sin(n\pi x)} \quad \left(\frac{2}{l} f(x)\right)$$

If $f(x) = \sum_{m=1}^{n} B_m \sin(m\pi x)$

then, $\left[u(x,t) = \sum_{m=1}^{n} B_m \sin(m\pi x) e^{-m^2\pi^2 t} \right]$

Any such f is continuous.

But $\cancel{f(x) \in [\,,}$ $\tilde{f}(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \in \{0,1\} \end{cases}$

Not continuous ; Hence, cannot be written using a finite sine series.

(One possible interpretation is that sol^n does not exist.)

$$f(x) \overset{??}{=} \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

$$u(x,t) \overset{?}{=} \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t},$$

How to find B_n's;

$$\int_0^1 \sin(m\pi x)\,.\,\sin(n\pi x) = \begin{cases} 0, & \text{if } m \neq n \\ 1/2, & \text{if } m = n \end{cases}$$

$$= \frac{1}{2}\,\delta_{mn}$$

$$\int_0^1 \sin(k\pi x)\, f(x)\, dx = \int_0^1 \left(\sum_{m=1}^{\infty} B_m \sin(m\pi x)\right) \sin(k\pi x)\, dx$$

$$= \sum_{m=1}^{\infty} B_m \int_0^1 \sin(m\pi x)\, \sin(k\pi x)\, dx$$

$$= B_k/2$$

$$\left[B_k = 2\int_0^1 f(x)\sin(k\pi x)\,dx \right]$$

Fourier observed that if we take large values of n on the ~~series~~ series

$\sum_{m=1}^{n} B_m \sin(m\pi x)$ (for B_m as defined above),

we can get functions that more closely approximate $f(x)=1$.

$$\left[1 = \sum_{m=1}^{\infty} \boxed{\frac{2}{m\pi}\left(1-(-1)^m\right)} \sin m\pi x \right]$$

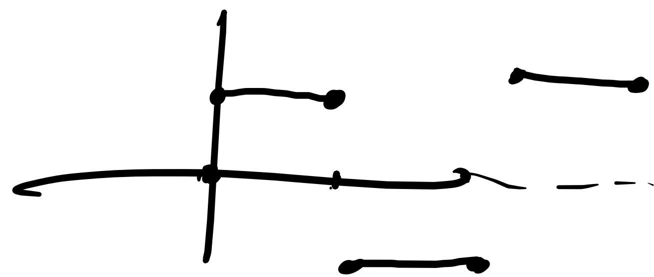
for $x \in (0,1)$

$$1 \Rightarrow \left(\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots \right)$$

(Gregory Madhavacharya series.)

$$f(x) \equiv \overbrace{\sum_{m=1}^{\infty} \frac{2}{m\pi}\left(1-(-1)^m\right) \underline{\underline{\sin(m\pi x)}}}$$

$$f(x+1) = -1 = -f(x)$$



$$\cancel{\sin(m\pi(x+1)) = \sin(m\pi x + m\pi)}$$

* Functions were polynomials; roots, powers, logarithms and whatever could be built up by addition, subtraction, multiplication, division, or composition of these functions.
* Functions had graphs with unbroken curves.
* Functions had derivatives and Taylor series.

$$1 = \sum_{m=1}^{\infty} a_m \cos((2m-1)\pi y)$$

Fourier's idea: Take derivatives w.r.t. y ($y=0$)

$$1 = \sum_{m=1}^{\infty} a_m$$

$$0 = \sum_{m=1}^{\infty} (2m-1)^2 a_m$$

$$0 = \sum_{m=1}^{\infty} (2m-1)^4 a_m$$

He solves the system for the first k equations and for $m > k$, he sets $a_m = 0$.

Using Cramer's rules he gets

$$a_1^{(k)} = \frac{3^2 \times 5^2 \times \cdots \times (2k-1)^2}{8 \times 24 \cdots \times (4k^2-4k)}$$

$$\lim_{k \to \infty} a_1^{(k)} = a_1 = \frac{4}{\pi}$$

$$a_m^{(k)} \longrightarrow (-1)^{m-1} \frac{4}{\pi(2m-1)}.$$

(infinite system of linear equations) $\boxed{A\vec{x} = \vec{b}}$

<u>Upto 1850</u>: Fourier's methods were being used to solve <u>PDE's</u>. Cauchy's ideas on convergence had been assimilated. (uniform convergence, uniform continuity)

Dirichlet, if f is piecewise monotonic on a closed and bounded interval, then the Fourier series converge to the original function.

FIVE big questions.

(1) When does a function have a Fourier series expansion that converges to that function?

(2) What is integration?

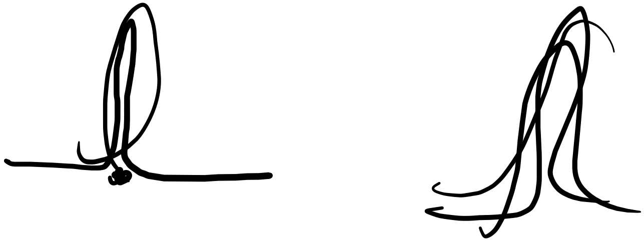
(3) What is the relationship between integration and differentiation?

(Lebesgue differentiation theorem)

$$\int_a^x f(t)\,dt$$

$$\frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} f \, dx = f \quad \text{a.e.}$$

(4) What is the relationship between continuity and differentiability?! (There are continuous functions on $\mathbb{R}$ which are nowhere differentiable.)



(5) When can an infinite series be integrated term-by term?

Qn 1. (a) Are the integrals that produce the Fourier coefficients well-defined? $\left(\int_0^1 \boxed{f(x)\sin(n\pi x)}\,dx\right)$

(b) If these integrals can be evaluated, then ~~the~~ does the resulting Fourier series converge to the original function (in an appropriate sense)?

~~Ex 4~~

(Study of <u>convergence</u> of functions,

(topologizing) function spaces)

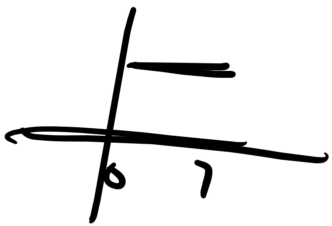
$E, F \to \boxed{\text{t.v.s.}}$

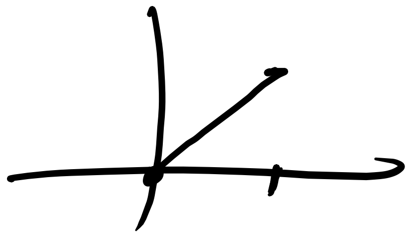
$u : \Omega \to F$

$\Omega \subseteq E.$

$C^1([0,1])$ $\quad \|f\| = \|f\|_\infty + \left\|\frac{df}{dx}\right\|_\infty$

The most interesting series (especially Fourier series) often do not converge uniformly, and yet term-by-term integration is valid.





Using Riemann integrals, justifying this is difficult.

As Lebesgue showed, there is a simple & elegant sol^{n}: DCT (dominated convergence theorem).

I look at the burning question of the foundations of infinitesimal analysis without sorrow, anger, or irritation. What Weierstrass – Cantor – did was *very good. That's the way it* had *to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justification" and their "proofs." One must be quite stupid not to see this at once, and quite careless if, after having seen this, one can get used to this artificial, logical atmosphere, and can later on forget this stark contradiction.*

– Nikolaĭ Nikolaevich Luzin

Classical ODE/PDE

integration (Lebesgue theory of measure)

convergence of functions

infinite system of linear eqns

integral equations

Hilbert space + Hilbert spectral theory

Metric spaces

Top. spaces

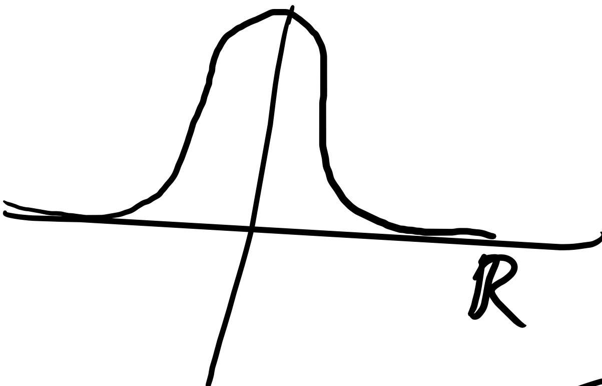
Normed sp..

Fréchet space

locally convex space

Distribution theory (generalized function differentiation)

Distribution theory → theory of generalized functions.



$C_c^\infty(\mathbb{R})$ → smooth functions with compact support.

$f_n \in C_c^\infty(\mathbb{R})$

$f_n \to f$

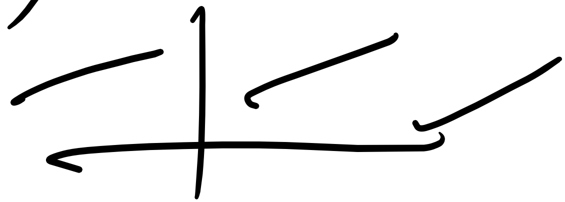
$\exists$ K compact, $\mathrm{supp}(f_n),\ \mathrm{supp}(f) \subseteq K$.

$\left\| \frac{d^n}{dx^n}(f_n - f) \right\|_\infty \to 0$ as $n \to \infty$

$u : C_c^\infty(\mathbb{R}) \to \mathbb{R}$

(distribution)

f



$\varphi_n \to \varphi$

$\frac{d\varphi_n}{dx} \to \frac{d\varphi}{dx}$

$$\langle u_f, \varphi \rangle = \int_{\mathbb{R}} f \varphi \, dx, \quad \varphi \in C_c^\infty(\mathbb{R})$$

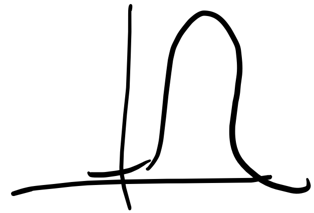
$$\boxed{u_f \longleftrightarrow f}$$

$$\int f_1 \varphi \, dx = \int f_2 \varphi \, dx$$

$$\left\langle \frac{du_f}{dx}, \varphi \right\rangle = \left\langle u_f, -\frac{d\varphi}{dx} \right\rangle$$

$\boxed{\frac{d}{dx} u_f}$ is a distribution.

derivative of f.



$u : \boxed{C_c^\infty(\mathbb{R})} \to \mathbb{R}$

linear functional on $C_c^\infty(\mathbb{R})$.

$L^2(\mathbb{R}) \subset \mathcal{D}(\mathbb{R})$ distributions

$H^s(\mathbb{R})$

$\int f \bar{g}$

$+\int \frac{df}{dx} \frac{\overline{dg}}{dx}$

$f \to \langle u_f, \varphi \rangle := \int_{\mathbb{R}} f\varphi \, dx$

$\boxed{f \subseteq u_f}$

$\langle u_f, \varphi_1 + \varphi_2 \rangle = \int f(\varphi_1 + \varphi_2) dx$

$= \int f\varphi_1 + \int f\varphi_2$

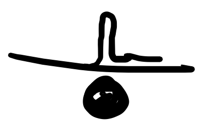
$u \in V^*$, $v \in V$

$\langle u, v \rangle = u(v)$

$f_1(x) \neq f_2(x)$

$$\int_{\mathbb{R}} f_1 \varphi \, dx = \int_{\mathbb{R}} f_2 \varphi \, dx$$

$$\int_{\mathbb{R}} (f_1 - f_2) \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R})$$



$$\int_{\mathbb{R}} \frac{df}{dx} \varphi \, dx = - \int_{\mathbb{R}} f \frac{d\varphi}{dx} \, dx$$

If f is diff.,

$$\left\langle u_{\frac{df}{dx}}, \varphi \right\rangle = \left\langle u_f, -\frac{d\varphi}{dx} \right\rangle$$

$$u : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$$

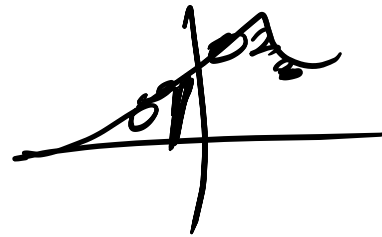
$$\left\langle \frac{du}{dx}, \varphi \right\rangle := \left\langle u, -\frac{d\varphi}{dx} \right\rangle$$

$$\frac{d}{dx} : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$$

PDE

First try to solve in the distributional sense.

$$\delta_0 : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$$



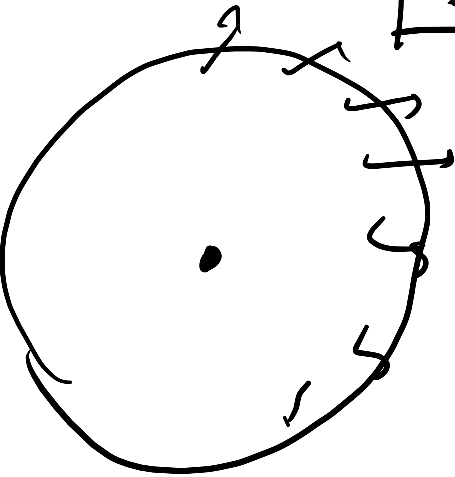
$$\langle \delta_0, \varphi \rangle = \varphi(0)$$

(Dirac distribution is not a function)

$$\left\langle \frac{\partial \delta_0}{\partial x}, \varphi \right\rangle = \cancel{-\varphi'(0)} \langle \delta_0, -\varphi' \rangle$$

$$= -\varphi'(0)$$

$$\boxed{\Delta u = \delta_0}$$



$(q)\ (q)$

$$\nabla \cdot \nabla u = \Delta$$

$(x_1^2+x_2^2+x_3^2)\ \hat{u}$

$= L$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

$= 0$ in charge-free

$$\hat{u} = \frac{1}{(x_1^2+x_2^2+x_3^2)}$$

$$\frac{1}{(x_1^2+\dots+x_n^2)^{1/2}} \qquad \frac{1}{r} \qquad \log$$