Functional Analysis - M. Math.

Assignment 7 — 2nd Semester 2021-2022

Due date: May 05, 2022 (by 11:59 pm)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) (a) (5 points) Let $\mathfrak{X}$ be a Banach space. Let $\rho : \mathfrak{X} \to \mathfrak{X}^{\#}$ be a linear map satisfying $\rho_x(x) \geq 0$ for all $x \in \mathfrak{X}$. Prove that ρ is a bounded operator.

   (b) (5 points) Let $C^1[0,1]$ denote the space of C^1-functions on $[0,1]$ endowed with the uniform norm inherited from $C[0,1]$. Let $D : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ be the differentiation operator. Prove that D is linear and has closed graph, but not continuous. (Deduce that $C^1[0,1]$ is not a closed subspace of $C[0,1]$.)

2. (20 points) (a) (5 points) Prove that there is **no** sequence of scalars $(c_n)_{n\in\mathbb{N}}$ with the property that a scalar series $(a_n)_{n\in\mathbb{N}}$ converges if and only if $a_n c_n \to 0$.

   (b) (5 points) Let $\mathfrak{X}$ be a Banach space and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{X}^{\#}$. Prove that the following assertions are equivalent:

      (i) For each norm-convergent series $\sum_{n=1}^{\infty} x_n$ in $\mathfrak{X}$, the scalar series $\sum_{n=1}^{\infty} \rho_n(x_n)$ converges;

      (ii) $\sum_{n=1}^{\infty} \|\rho_n - \rho_{n+1}\| < \infty$.

   (c) (5 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Show that the series $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges if and only if the series $\sum_{n=1}^{\infty} a_n b_n$ converges for every convergent series $\sum_{n=1}^{\infty} b_n$ in $\mathbb{R}$.

   (d) (5 points) Prove that a scalar series $\sum_{n=1}^{\infty} a_n$ converges if and only if the scalar series

$$\sum_{n=1}^{\infty} \Big(1 - \frac{1}{2^2} + \cdots + \frac{(-1)^n}{n^2}\Big) a_n,$$

   converges.

   (Hint: Use the uniform boundedness principle in the appropriate setting.)

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3. (10 points) A subset Y of a Hilbert space $\mathscr{H}$ is said to be an *orthonormal set* if it consists of mutually orthogonal unit vectors. If Y is an orthonormal set in $\mathscr{H}$, prove that the following conditions are equivalent:
   (i) $[Y] = \mathscr{H}$ (that is, the linear span of Y is dense in $\mathscr{H}$);
   (ii) For each $u \in \mathscr{H}$, $u = \sum_{y\in Y} \langle u, y\rangle y$;
   (iii) For each $x \in \mathscr{H}$, $\|u\|^2 = \sum_{y\in Y} |\langle u, y\rangle|^2$;
   (iv) For each $u, v \in \mathscr{H}$, $\langle u, v\rangle = \sum_{y\in Y} \langle u, y\rangle\langle y, v\rangle$.

4. (20 points) Let $\mathbb{D}$ be the open unit disc of radius 1 centred at the origin of the complex plane. The Bergman space, $L^2_h(\mathbb{D})$, consists of those homolorphic functions on $\mathbb{D}$ that are square-summable with respect to the planar Lebesgue measure, m. Consider the inner product on $L^2_h(\mathbb{D})$ given by

$$\langle f, g\rangle_{L^2_h(\mathbb{D})} = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}\, dm(z).$$

   (a) (5 points) Show that $(L^2_h(\mathbb{D}); \langle \cdot, \cdot\rangle_{L^2_h(\mathbb{D})})$ is a Hilbert space.
   (b) (5 points) For $\lambda \in \mathbb{D}$, show that the evaluation map $\rho_\lambda : L^2_h(\mathbb{D}) \to \mathbb{C}$ given by $\rho_\lambda(f) = f(\lambda)$, is continuous. Use the Riesz representation theorem - for Hilbert spaces - to conclude that there is a (unique) function $K_\lambda \in L^2_h(\mathbb{D})$ such that $f(\lambda) = \langle f, K_\lambda\rangle$ for all $f \in L^2_a(\mathbb{D})$. (The function K_λ is called the *reproducing kernel* at λ because it reproduces the value of any function at λ.)
   (c) (5 points) Prove that $Y := \{\sqrt{n+1}z^n : n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal set in $L^2_h(\mathbb{D})$ whose span is dense in $L^2_h(\mathbb{D})$.
   (d) (5 points) Show that $\langle K_\lambda, K_\mu\rangle = \sum_{n=0}^{\infty}(n+1)\lambda^n\overline{\mu}^n = (1 - \lambda\overline{\mu})^{-2}$.
   (Hint: Use Parseval's formula from Problem 3, and the orthonormal basis from part (ii) of this problem.)
   In other words, $K_\lambda(z) = \langle K_\lambda, K_z\rangle = (1 - \lambda\overline{z})^{-2}$ for all $z \in \mathbb{D}$. Thus we have explicitly computed the reproducing kernel function at λ and we have the formula,

$$f(\lambda) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)(1 - \overline{\lambda}z)^{-2}\, dm(z), \forall \lambda \in \mathbb{D}.$$

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