Functional Analysis - M. Math.

Assignment 6 — 2nd Semester 2021-2022

Due date: April 22, 2022 (by 11:59 pm)
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

For a compact Hausdorff space X, we use the notation $C(X)$ to denote the Banach space of complex-valued continuous functions on X with the *sup* norm.

1. (10 points) Let $\mathfrak{X}$ be a Banach space and $\mathfrak{X}_0$ be a closed subspace of $\mathfrak{X}$. A vector $x \in \mathfrak{X}$ is said to be ε-perpendicular to $\mathfrak{X}_0$ if $\|x + y\| \geq (1 - \varepsilon)\|x\|$ for any $y \in \mathfrak{X}_0$.
   (a) (5 points) Prove that for $\varepsilon > 0$, any proper closed subspace $\mathfrak{X}_0$ has an ε-perpendicular in $\mathfrak{X}$.
   (b) (5 points) Using the above result, show that the unit-ball of an infinite-dimensional Banach space is not compact (with respect to the norm topology).

2. (10 points) Consider $L := \{f \in C[-1, 1] : \int_{-1}^{0} f(x)dx = \int_{0}^{1} f(x)dx\}$.
   (a) (5 points) Prove that L is a closed linear subspace of $C[-1, 1]$ and find $\rho \in (C[-1, 1])^{\#}$ such that $L = \ker \rho$.
   (b) (5 points) Prove that ρ from the above part does not attain its norm on the closed unit ball.

3. (10 points) Let $\mathfrak{X}$ be a Banach space.
   (a) (5 points) Prove that the set of norm-attaining functionals in $\mathfrak{X}^{\#}$ is norm-dense in $\mathfrak{X}^{\#}$.
   (b) (5 points) Prove that $\mathfrak{X}$ is reflexive if and only if $\mathfrak{X}^{\#}$ is reflexive.

4. (10 points) Let X be a compact Hausdorff space and let $\mathscr{D}(X) := \{\delta_x : x \in X\} \subseteq (C(X))^{\#}$ denote the space of Dirac measures on X. We have seen in class that $\mathscr{D}(X)$ is the set of extreme points of the set of regular Borel probability measures on X.
   (a) (5 points) Show that $\mathscr{D}(X)$ with the weak-$*$ topology is homeomorphic to X.
   (b) (5 points) Show that every regular Borel probability measure on X is the weak-$*$ limit of finitely supported probability measures (discrete probability measures) on X.

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5. (20 points) A smooth function $f : (0,\infty) \to \mathbb{R}$ is said to be *completely monotonic* if $(-1)^n f^{(n)} \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Let BCM denote the set of bounded and completely monotonic functions on $(0,\infty)$. *The goal of this exercise is to show that every element of BCM is the Laplace transform of a finite positive Borel measure on* $[0,\infty)$.

   Consider the topology on $C^\infty(0,\infty)$ given by the (separating) family of seminorms (for $m, n \in \mathbb{N}$):
$$p_{m,n}(f) = \sup\{|f^{(k)}(x)| : x \in [\frac{1}{n}, n], 0 \leq k \leq m\}.$$
   Note that $C^\infty(0,\infty)$ is metrizable (as the family of seminorms is countable). We endow BCM with the corresponding subspace topology.

   We define
$$K := \{f \in BCM : f(0^+) \leq 1\},$$
   which is a convex subset of BCM, and denote the set of extreme points of K by $\text{ext}(K)$.

   (a) (5 points) For $x_0 > 0$ and $f \in K$, let $u_{f,x_0}(x) := f(x+x_0) - f(x_0)f(x)$. Show that $f + u_{f,x_0}, f - u_{f,x_0} \in K$. Conclude that every element of $\text{ext}(K)$ is of the form $f(x) = e^{-\alpha x}$ for some $\alpha \in [0,\infty]$ with the convention that $e^{-\infty x} = 0$. (Hence, $\text{ext}(K) \subseteq \{e^{-\alpha x} : \alpha \in [0,\infty]\}$.)

   (b) (5 points) For $\beta > 0$, consider the (dilation) transformation $T_\beta : C^\infty(0,\infty) \to C^\infty(0,1)$ given by $T_\beta(f)(x) = f(\beta x)$ for $x > 0$. Prove that $T_\beta(K) \subseteq K$, and $T_\beta(ext(K)) = ext(K)$. (Conclude that $\text{ext}(K) = \{e^{-\alpha x} : \alpha \in [0,\infty]\}$.)

   (c) (5 points) Prove that K and $\text{ext}(K)$ are compact subsets of BCM. (Hint: Recall that BCM is metrizable and prove sequential compactness.)

   (d) (5 points) Show that for every $f \in BCM$ there is a finite regular Borel measure on $[0,\infty)$ such that
$$f = \int_0^\infty e^{-\alpha x} d\mu(\alpha).$$
   (Hint: Here you may assume the Riesz representation theorem for compact Hausdorff spaces, that is, $C[0,\infty]^\#$ is isometrically isomorphic to the space of regular Borel measures on $[0,\infty]$.)

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