Functional Analysis - M. Math.

Assignment 5 — 2nd Semester 2021-2022

Due date: March 31, 2022 (by 11:59 pm)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) Prove that the following properties of a subset A in a topological space X are equivalent:

   (i) A is compact;

   (ii) every infinite subset of A contains a net converging to some element of A;

   (iii) every centred system of closed subsets of A has non-empty intersection. (A system of sets is said to be *centred* if any finite subsystem has non-empty intersection, that is, it has the finite intersection property.)

2. (10 points) Let V be a (real or complex) vector, on which two locally convex topologies, τ_1 and τ_2, are specified. Let Γ_j be the set of all τ_j-continuous semi-norms on V, for $j = 1, 2$; and let

$$\Gamma := \{q_1 + q_2 : q_1 \in \Gamma_1, q_2 \in \Gamma_2\}.$$

   (a) (5 points) Show that Γ is a separating family of semi-norms on V, and that the corresponding locally convex topology τ is the coarsest locally convex topology on V that is finer than both τ_1 and τ_2.

   (b) (5 points) Show that a linear functional ρ on V is τ-continuous if and only if it has the form $\rho_1 + \rho_2$, where ρ_j is a τ_j-continuous linear functional on V, for $j = 1, 2$.

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3. (20 points) (a) (5 points) For $n \in \mathbb{N}$, consider $\mathbb{R}^n$ with the ℓ^p-norm ($\|\cdot\|_p$) for $1 \leq p \leq \infty$. Find all extreme points of the unit ball of the normed linear space $(\mathbb{R}^n; \|\cdot\|_p)$.

   (b) (5 points) Let c be the set of all sequences $\{x_n\}$ for which $\lim_{n\to\infty} x_n$ exists, endowed with the norm $\|\{x_{\}}\| := \sup |x_n|$. Find all extreme points of the unit ball of c.

   (c) (5 points) Let c_0 be the set of all sequences $\{x_n\}$ for which $\lim_{n\to\infty} x_n = 0$, endowed with the norm $\|\{x_{\}}\| := \sup |x_n|$. Find all extreme points of the unit ball of c_0.

   (d) (5 points) Find all extreme points of the set S of doubly stochastic $n \times n$ matrices. (A matrix A with real entries is said to be *doubly stochastic* if its elements are non-negative, and the sum of elements in each row and in each column is equal to 1.)

4. (10 points) Let $x_1, \ldots, x_n$ be elements of a normed $\mathbb{K}$-linear space $\mathfrak{X}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that the closure of the set,

$$\{a_1 x_1 + \cdots + a_n x_n : a_j \in \mathbb{K}, \prod_{j=1}^{n} (a_j - 1) = 0\},$$

   contains $\vec{0}$. Show that $x_1, \ldots, x_n$ are linearly dependent.

5. (10 points) Suppose Y and Z are closed subspaces of a Banach space $\mathfrak{X}$, such that $Y \cap Z = \{\vec{0}\}$, and let

$$\alpha := \inf\{\|y - z\| : y \in Y, z \in Z, \|y\| = \|z\| = 1\}.$$

   Show that the subspace $Y + Z$ is closed if and only if $\alpha > 0$.

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