Functional Analysis - M. Math.

Assignment 4 — 2nd Semester 2021-2022

Due date: March 24, 2022 (by 11:59 pm)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (15 points) Prove the following assertions.
   (a) (5 points) If V is a finite-dimensional vector space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), there is a unique locally convex topology on V.
   (b) (5 points) A locally convex space V is locally compact if and only if V is finite-dimensional.
   (c) (5 points) If V_0 is a finite-dimensional subspace of a locally convex space V, then V_0 is closed in V.

   (Hint: These are results from §1.2 in Kadison-Ringrose Volume I.)

2. (10 points) Let V be a vector space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). Prove that among all local convex topologies on V, there is a finest topology $\mathcal{T}_{cc}$ (which is called the *convex core topology*), and that it has the following properties:
   (a) (5 points) Every linear functional $\rho : (V, \mathcal{T}_{cc}) \to \mathbb{K}$ is continuous; Equivalently, every hyperplane (subspace of codimension one) is closed in $(V, \mathcal{T}_{cc})$.
   (b) (5 points) A basis of neighbourhoods of zero in $(V, \mathcal{T}_{cc})$ consists of all the convex sets which contain zero and are comprised entirely of internal points.

   (Recall that in class, we first proved the Hahn-Banach theorems without making any overt appeal to topologies on V by just considering the notion of internal points. In this problem, we show that the notion of internal points and convexity naturally endows V with a topological structure where *internal points* become *interior points*. Thus there is no loss of generality in viewing the Hahn-Banach theorems purely in the TVS setting.)

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3. (10 points) Let V be a vector space over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$. For a subset K of V, the *Minkowski functional* of K is defined to be the function $p_K : V \to \mathbb{R} \cup \{\infty\}$, valued in the extended real numbers, given by

$$p_K(\vec{x}) := \inf\{c \in \mathbb{R} : c > 0 \text{ and } \vec{x} \in cK\}, \text{ for every } \vec{x} \in V,$$

where the infimum over the empty set is defined to be ∞.

   (a) (5 points) Prove that among all the convex sets B in V containing 0 and having a given Minkowski functional p, there is a largest B_1 and a smallest B_0 with respect to set inclusion.

   (b) (5 points) Show that B_1 is the closure of B_0 in the *convex core topology*, defined in the previous problem.

4. (15 points) A set M in a TVS is said to be *bounded* if for each neighborhood U of zero, there is an $\varepsilon > 0$ such that $\varepsilon M \subset U$.

   Let $\mathbb{R}^\omega$ be the space of all sequences of real numbers with the product topology (or equivalently, the topology of pointwise convergence).

   (a) (5 points) Show that $\mathbb{R}^\omega$ is metrizable, with the metric $d : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}_{\geq 0}$ defined by

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

   recovering the topology of pointwise convergence.

   (b) (5 points) Prove that $\mathbb{R}^\omega$ is locally convex.

   (c) (5 points) Prove that there are no non-empty open bounded sets in $\mathbb{R}^\omega$. Conclude that $\mathbb{R}^\omega$ is not normable.

5. (10 points) Prove that every proper closed convex set in a real LCS V is the intersection of some family of half-spaces of the form $\{\vec{x} \in V : \rho(\vec{x}) \leq c\}$, where ρ is a continuous linear functional on V and $c \in \mathbb{R}$.

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