Functional Analysis - M. Math.

Assignment 3 — 2nd Semester 2021-2022

Due date: March 06, 2022 (by 11:59 pm)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (6 points) ((Supporting hyperplane theorem)) Let V be a real vector space and K be a proper non-empty convex subset of V with an internal point. Let x be a point in K such that for some vector $v \in V \backslash \{0\}$ the line segment $[x - \varepsilon v, x + \varepsilon v]$ is not a subset of K for all $\varepsilon > 0$. Show that there is a hyperplane H passing through x such that K is contained in one of the "closed" halfspaces determined by H. (Note that for $V = \mathbb{R}^n$ with the usual topology, such points x would be boundary points of K.)

2. (14 points) Let $n \in \mathbb{N}$. For a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the *support function* $h(K, \cdot) = h_K : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$h(K, \vec{u}) := \sup\{\langle \vec{x}, \vec{u} \rangle \mid x \in K\} \text{ for } \vec{u} \in \mathbb{R}^n.$$

   (a) (6 points) Show that the range of h_K is in $\mathbb{R}$ if and only if K is compact.

   (b) (8 points) If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique non-empty compact convex set K in $\mathbb{R}^n$ such that $f = h_K$. (Hint: Show that the epigraph of f is a closed convex cone in $\mathbb{R}^n \times \mathbb{R}$. Use Problem 1 at $(\vec{0}, 0)$.)

Soumyashant Nayak: soumyashant@gmail.com, soumyashant@isibang.ac.in

3. (10 points) If K is a subset of a real or complex vector space V, then the *Minkowski functional* of K is defined to be the function $p_K : V \to \mathbb{R} \cup \{\infty\}$, valued in the extended real numbers, given by

$$p_K(\vec{x}) := \inf\{c \in \mathbb{R} : c > 0 \text{ and } \vec{x} \in cK\}, \text{ for every } \vec{x} \in V,$$

where the infimum over the empty set is defined to be ∞.

Let q be a sublinear functional on a (real or complex) vector space V, and

$$K := \{x \in V : q(x) < 1\}.$$

(a) (3 points) Show that K is convex, contains 0 and consists entirely of internal points.

(b) (7 points) Show that the Minkowski functional of K is given by

$$p(x) = \max\{q(x), 0\} \text{ for all } x \in V.$$

4. (10 points) Suppose that q_1 and q_2 are semi-norms on a (real or complex) vector space V, ρ is a linear functional on V, and

$$|\rho(x)| \leq q_1(x) + q_2(x) \text{ for all } x \in V.$$

Show that ρ can be expressed in the form $\rho_1 + \rho_2$, where ρ_1, ρ_2 are linear functionals on V, and

$$|\rho_1(x)| \leq q_1(x), |\rho_2(x)| \leq q_2(x) \text{ for all } x \in V.$$

5. (12 points) Let α be a cardinal number, and the product of lines $\mathbb{R}^\alpha$ be equipped with the product topology.

(a) (6 points) Show that $\mathbb{R}^\alpha$ is a locally convex space. Provide a family of seminorms on $\mathbb{R}^\alpha$ which determines the product topology on $\mathbb{R}^\alpha$.

(b) (6 points) Prove that a given locally convex space can be imbedded continuously in $\mathbb{R}^\alpha$ for a sufficiently large cardinal number α. (In other words, every LCS admits a coordinate description.)

6. (8 points) Let $M := [0,1]^{[0,1]}$ denote the space of functions from $[0,1]$ into itself. For $x \in [0,1]$, note that $p_x(f) = |f(x)|$ defines a seminorm on M.

(a) (3 points) Show that $\{p_x : x \in [0,1]\}$ defines a separating family of seminorms on M, and hence a locally convex topology $\mathcal{T}$ on M.

(b) (5 points) Show that $\mathcal{T}$ is not metrizable.

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