Functional Analysis - M. Math.

Assignment 2 — 2nd Semester 2021-2022

Due date: February 25, 2022 (by 11:59 pm)
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

Throughout the assignment, μ is a positive measure on the measure space X.

1. (20 points) A sequence $\{f_n\}$ of complex measurable functions on X is said to converge *locally in measure* to the measurable function f if for every $\varepsilon > 0$ and every measurable set F with $\mu(F) < \infty$, there corresponds an $N \in \mathbb{N}$ such that

$$\mu(\{x \in F : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n > N$. Prove the following statements:

(a) (5 points) If $f_n(x) \to f(x)$ a.e., then $f_n \to f$ locally in measure.

(b) (5 points) If $f_n \in L^p(X;\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ locally in measure; here $1 \leq p \leq \infty$.

(c) (5 points) If $f_n \to f$ locally in measure, then $f_n^2 \to f^2$ locally in measure.

(d) (5 points) If $f_n \to f$ locally in measure, then $\{f_n\}$ has a subsequence which converges to f a.e.

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2. (30 points) We say that μ has a *countable base* if there exists a countable family $\{A_n\}$ of measurable subsets of X such that for any measurable subset B there is a set A_n for which $\mu(A_n \Delta B) < \varepsilon$. (Here $A\Delta B$ denotes set symmetric difference, $A\backslash B \cup B\backslash A$.)
   (a) (5 points) Prove that $L^1(X;\mu)$ is a separable space if and only if μ has a countable base.
   (b) (5 points) Prove that $L^p(X;\mu)$, $1 < p < \infty$, is a separable space if and only if $L^1(X;\mu)$ is a separable space.
   (c) (5 points) Prove that $L^\infty(X;\mu)$ is either finite-dimensional or non-separable.
   (d) (5 points) Assuming that $\mu(X) < \infty$, prove that for $1 \le p \le q$, the space $L^q(X;\mu)$ is contained in the space $L^q(X;\mu)$.
   (e) (5 points) Prove that, if $1 \le p < q \le \infty$, then neither of the spaces $L^p(\mathbb{R};m)$, $L^q(\mathbb{R};m)$ is contained in the other.
   (f) (5 points) Let $0 < \alpha \le \beta < \infty$. For what values of p in $[1,\infty]$ does the function
   $$\frac{1}{x^\alpha + x^\beta},$$
   belong to $L^p(\mathbb{R}_+;m)$?

3. (10 points) Let $1 \le p \le r \le q \le \infty$. Prove that for every measurable function f, we have
   $$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^\beta,$$
   where
   $$\alpha = \frac{r^{-1} - q^{-1}}{p^{-1} - q^{-1}}, \beta = \frac{p^{-1} - r^{-1}}{p^{-1} - q^{-1}}.$$
   Conclude that $L^p(X;\mu) \cap L^q(X;\mu)$ is contained in $L^r(X;\mu)$.
   (Hint: Note that $\frac{1}{r} = \frac{\alpha}{p} + \frac{\beta}{q}, \alpha + \beta = 1$.)

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