Functional Analysis - M. Math.

Assignment 1 — 2nd Semester 2021-2022

Due date: February 14, 2022 (by 11:59 pm)
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

We denote the set of $m \times n$ matrices with real entries by $M_{m,n}(\mathbb{R})$. When $m = n$, we use the notation $M_m(\mathbb{R})$.

Let X be a measure space.

1. (10 points) (a) (5 points) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x \in X : f(x) < g(x)\}, \{x \in X : f(x) = g(x)\},$$

are measurable.

(b) (5 points) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

2. (10 points) Suppose μ is a positive measure on X, $f : X \to [0, \infty)$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and $\alpha > 0$ is a constant. Evaluate the following limit (with justification):

$$\lim_{n\to\infty} \int_X n \log\left(1 + (f/n)^{\alpha}\right) d\mu.$$

3. (10 points) Suppose $f \in L^1(X; \mu)$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

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4. (20 points) Let $f \in L^1(\mathbb{R}; m)$ where m denotes the Lebesgue measure on $\mathbb{R}$. The Fourier transform of f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined as follows:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dm(x).$$

   (a) (6 points) (Riemann-Lebesgue lemma) Show that $\widehat{f}$ is a continuous function on $\mathbb{R}$ vanishing at infinity.

   (b) (4 points) Show that $\widehat{f}(\xi) \leq \|f\|_1$ for all $\xi \in \mathbb{R}$. In other words, $\|\widehat{f}\|_\infty \leq \|f\|_1$.

   (c) (5 points) Let $g(x) = xf(x)$ for $x \in \mathbb{R}$. If $g \in L^1(\mathbb{R}; m)$, show that $\widehat{f}$ is differentiable with

$$\frac{d}{d\xi}(\widehat{f}) = (-2\pi i)\widehat{g}.$$

   (d) (5 points) Let $g(x) = x^n f(x)$ for $x \in \mathbb{R}$. If $g \in L^1(\mathbb{R}; m)$, show that $\widehat{f}$ is differentiable n times with

$$\frac{d^n}{d\xi^n}(\widehat{f}) = (-2\pi i)^n \widehat{g}.$$

(Loosely speaking, the Fourier transform converts "multiplication" by a polynomial function into "differentiation" by the corresponding differential operator.)
Hint: Parts (c), (d) involve usage of the dominated convergence theorem.

5. (10 points) Let $f \in C_c^n(\mathbb{R})$ Note that $f \in L^1(\mathbb{R}; m)$ and so does its nth order derivative. Let $g = \frac{d^n}{dx^n} f$.

   (a) (5 points) Prove that

$$\widehat{g}(\xi) = (2\pi i \xi)^n \widehat{f}(\xi).$$

   (Loosely speaking, the Fourier transform converts a polynomial differential operator into "multiplication" by the corresponding polynomial function.)

   (b) (5 points) Show that there is a constant $C > 0$ such that

$$|\widehat{f}(\xi)| \leq \frac{C}{(1 + |\xi|^n)} \forall \xi \in \mathbb{R}.$$

   (Loosely speaking, the Fourier transform of a C_c^n function decays faster than the function $\frac{1}{|\xi|^n}$ near infinity.)
   Hint: Use question 4, (b).

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