13/08/2021   **Lecture - 9**

Theorem. Suppose $\varphi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$.

$\left(\boxed{\varphi_\varepsilon(x)} = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right), \text{ for } \varepsilon > 0\right)$.

If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ $\boxed{\text{or } f \in C_0(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)}$

then, $\|f * \varphi_\varepsilon - f\|_p \to 0$ as $\varepsilon \to 0$.

$(1 \le p \le \infty)$.

Pf: $(f * \varphi_\varepsilon)(t) - f(t) = \int_{\mathbb{R}^n} \boxed{\left(f(t - x) - f(t)\right)} \varphi_\varepsilon(x)\,dx$

$$\Rightarrow \|f * \varphi_\varepsilon - f\|_p \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(t-x) - f(t)|^p \varphi_\varepsilon(t)\, dt \right) dx.$$

$$\|\tau_x f - f\|_p \le \|\tau_x f - \tau_x g\|_p + \|\tau_x g - g\|_p + \|g - f\|_p.$$

$$\boxed{g \in C_c(\mathbb{R}^n)}$$

Corollary of Multiplication Formula

If $f, \varphi \in L^1(\mathbb{R}^n)$, and $\varphi = \hat{\Phi}$, then

$(\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon}))$.

$$\boxed{\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i \langle x, \xi\rangle}\, \boxed{\Phi(\varepsilon x)}\, dx}$$

Φ mean of $\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i \langle x, \xi\rangle} dx$

$$= \int_{\mathbb{R}^n} f(x)\, \varphi_\varepsilon(x - \xi)\, dx$$

$$= \boxed{(f * \tilde{\varphi}_\varepsilon)(\xi)}.$$

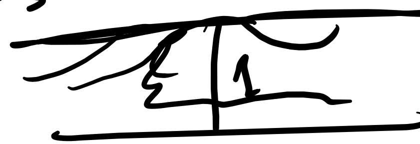
Theorem (Fourier inversion formula in the L^1-sense)

If Φ and its Fourier transform $\varphi = \hat{\Phi}$ are integrable, and $\left(\int_{\mathbb{R}^n} \varphi(x)\, dx = 1\right)$, then

$\Phi(0)$

the Φ-mean of $\left[\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, d\xi\right]$ (given by

$\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, \boxed{\Phi(\varepsilon \xi)}\, d\xi$) converges to f in the L^1-norm.



Proof: Use multiplication formula:

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i\langle x,\xi\rangle}\, \Phi(\varepsilon\xi)\, d\xi$$

$$= (f * \tilde{\Phi}_\varepsilon)(x).$$

$\longrightarrow f$ in the L^1 sense.

Corollary: If both f, $\hat{f} \in L^1(\mathbb{R}^n)$, then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i\langle x,\xi\rangle}\, d\xi$$ for almost every $x \in \mathbb{R}^n$.

Proof. Choose Φ to be a Schwarz function with $\Phi(0) = 1$.

$$f * \tilde{\Phi}_{\varepsilon} \longrightarrow f \text{ in the } L^1\text{-sense}$$

There is a sequence $a_k \to 0$ such that $f * \tilde{\Phi}_{a_k} \longrightarrow f$ pointwise a.e.

Use DCT (since $\hat{f} \in L^1$).

$$\lim_{a_k \to 0} \Phi(a_k x) = 1, \quad \forall x \in \mathbb{R}^n.$$

and thus

$$\boxed{f(x) = \int_{\mathbb{R}^n} \underbrace{\hat{f}(\xi)}_{\geq 0} e^{2\pi i\langle x,\xi\rangle}\, d\xi}$$

for almost every $x \in \mathbb{R}^n$.

Corollary: If $f_1, f_2 \in L^1(\mathbb{R}^n)$, and $\hat{f}_1(x) = \hat{f}_2(x)$ for all $x \in \mathbb{R}^n$, then

$f_1 = f_2$ a.e.

Proof Let $f \in L^1(\mathbb{R}^n)$, and $\hat{f} = 0$.

Then $f = 0$ a.e.

$\hat{f}_1 = \hat{f}_2$,

$f_1 - f_2 \in L^1(\mathbb{R}^n)$, $\widehat{f_1 - f_2} = 0$.

Fourier transform on $L^1(\mathbb{R}^n)$ is injective.

Theorem: (Fourier inversion formula in the pointwise sense)

Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$.

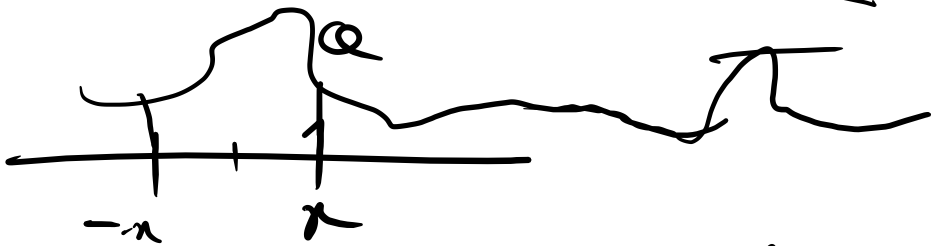
If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then

$$\lim_{\varepsilon \to 0} (f * \varphi_\varepsilon)(x) = f(x)\left(\int_{\mathbb{R}^n} \varphi(t)\, dt\right)$$

whenever x belongs to the Lebesgue set of f.

(Fejér-Lebesgue term... $\sigma_n(f)(x) = f(x)$).

(We just need, $\Psi(x) := \operatorname*{ess\,sup}_{|t| \geq |x|} |\varphi(t)|$



)

Ψ is a decreasing radial function.

We need $\Psi \in L^1(\mathbb{R}^n)$]

Let $a := \int_{\mathbb{R}^n} \varphi_\varepsilon(t)\, dt = \int_{\mathbb{R}^n} \varphi(t)\, dt \quad (\forall \varepsilon > 0)$.

and $x \in$ Lebesgue set of f.

For every $\delta > 0$, $\exists\, \rho > 0$ such that

$$\boxed{\frac{1}{r^n} \int_{|t|<r} |f(x-t) - f(x)|\, dt} < \delta,$$

for $r \leq \rho$. -

$$\left| (f * \alpha_\varepsilon)(x) - a f(x) \right|$$

$$= \left| \int_{\mathbb{R}^n} (f(x-t) - f(x))\, \alpha_\varepsilon(t)\, dt \right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-t) - f(x)|\, |\alpha_\varepsilon(t)|\, dt$$

$$\leq \int_{|t|\leq \rho} |(f(x-t) - f(x))||\varphi_\varepsilon(t)|\,dt$$

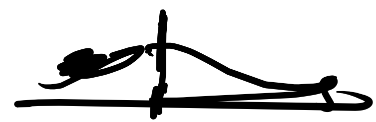
$\hookrightarrow I_1$

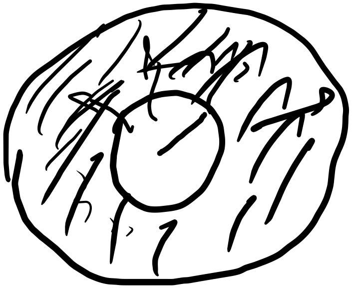
$$+ \int_{|t|\geq \rho} |(f(x-t) - f(x)||\varphi_\varepsilon(t)|\,dt$$

$\hookrightarrow I_2$

<u>Estimating I_1:</u> Note that $\psi(x)$ is radial (that is, $\psi(x_1) = \psi(x_2)$ if $|x_1| = |x_2|$). $(\psi : \mathbb{R}^n \to \mathbb{R})$

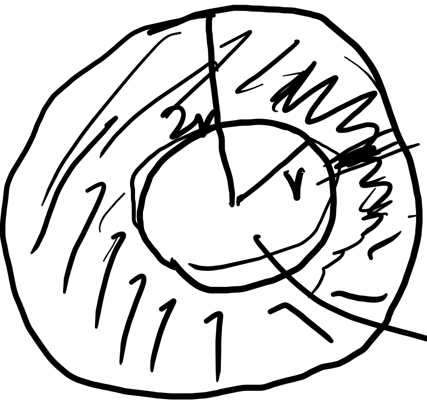
If $\psi_0(r) := \psi(x)$, when $|x| = r$, then ψ_0 is a decreasing function of r





$$\boxed{\frac{\Omega_n (2^n - 1)}{2^n} r^n \psi_0(r)} \to \psi(x)$$

$$\leq \int_{\boxed{\frac{r}{2} \leq |x| \leq r}} \psi(x)\, dx \to 0 \text{ as } r \to \infty$$



vol. $\Omega_n r^n$

vol. of ball of radius $2r \to \Omega_n (2r)^n$

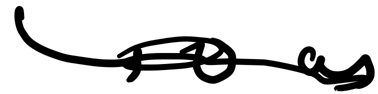
$= 2^n \Omega_n r^n$

vol. of n-dim annulus

$= \Omega_n (2^n - 1) r^n$.

$$= \hat{\psi}_0(r) \leq \frac{1}{\operatorname{Vol}(2^{k-1}(\frac{r}{2})^n)} \int_{\frac{r}{2} \leq |x| \leq r} \psi(x)\,dx$$

ψ is rather large



Proof.

$$\lim_{r\to\infty} \int_{\frac{r}{2} \le |x| \le r} \psi(x)\,dx = 0$$

(by DCT)

$$\boxed{\frac{\Omega_n(2^n-1)}{2^n}}\ \boxed{r^n \psi_0(r)} \to 0 \text{ as } r \to \infty$$

$\to 0$ as $r \to 0$

$$\left[\ \psi_0(r) \le \frac{A}{r^n} \quad \text{for } 0 < r < \infty,\ \right]$$

$$g(r) := \int_{S^{n-1}} | f(x - rt') - f(x) | \, dt'$$

($t' \to$ element of surface area of S^{n-1}).

$$G(r) := \int_0^r \boxed{s^{n-1} g(s)} \, ds \le \delta r^n \quad (\text{for } r \le \rho).$$

$\to$ be

$| \varphi_\varepsilon(t) = \frac{1}{\varepsilon^n} \varphi\left(\frac{t}{\varepsilon}\right)$

$$I_1 \le \int_{|t| < \rho} | f(x-t) - f(x) | \, \varepsilon^{-n} \psi\left(\frac{t}{\varepsilon}\right) dt$$

$| \varphi(x) | \le \psi(x)$.

$$= \int_0^\rho \boxed{r^{n-1}} \, g(r) \, \varepsilon^{-n} \psi_0\left(\frac{r}{\varepsilon}\right) dr.$$

$$= G(r)\, \varepsilon^{-n} \psi_0\left(\frac{r}{\varepsilon}\right)\Big]_0^{\rho}$$

$$- \int_0^{\rho} G(r)\, \underline{d}\left(\varepsilon^{-n} \psi_0\left(\frac{r}{\varepsilon}\right)\right)$$

$(r^n \psi_0(r) \leq A)$

$$\leq \delta \rho^n \varepsilon^{-n} \psi_0(\rho/\varepsilon)$$

$$- \int_0^{\rho/\varepsilon} \underline{\underline{G(\varepsilon s)}}\, \varepsilon^{-n}\, \underline{\underline{d}}\psi_0(s)$$

(ψ_0 - monotonically decreasing)

$$\leq \delta A - \int_0^{\rho/\varepsilon} \delta s^n \, d\psi_0(s).$$

$$\le \delta \left(A + \left(- \int_0^\infty s^n \, d\psi_0(s) \right) \right.$$

$$\boxed{r^n \psi_0(r)} \to 0 \text{ as } r \to 0, \; r \to \infty$$

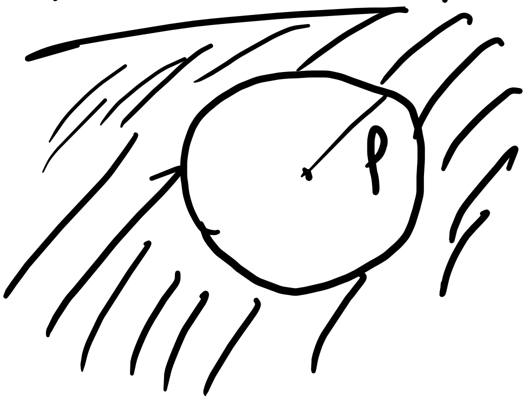
$$\boxed{n \int_0^\infty s^{n-1} \, d\psi_0(s) \, ds}$$

$$\boxed{\int f g' + \int g f' = fg \Big]_{\text{bounds}}}$$

$$\frac{n}{|S^{n-1}|} \int_{\mathbb{R}^n} \psi(x)\, dx$$

$|S^{n-1}|$: surface area of unit sphere

Estimating I_2



$\chi_\rho \to$ indicator function of $|x| \geq \rho$.

$$\frac{1}{r^n} \int_{|t|<\rho r} |f(x-t) - f(x)|\, dt < \delta$$

$\forall \; r \leq \rho$.

$$I_2 = \left| \int_{|t| \geq \rho} (f(x-t) - f(x))\, a_\varepsilon(t)\, dt \right|$$

$$I_2 \le \left| \int_{|t| \ge \rho} f(x-t)\, \varphi_\varepsilon(t)\, dt \right|$$

$$+ \left| \int_{|t| \ge \rho} |f(x)|\, \varphi_\varepsilon(t)\, dt \right|$$

$$\le \|f\|_p \, \|\chi_\rho \varphi_\varepsilon\|_q + \boxed{|f(x)| \cdot \|\chi_\rho \varphi_\varepsilon\|_1}$$

$$\|\chi_\rho \varphi_\varepsilon\|_1 = \int_{|x| \ge \rho} \underbrace{\varphi_\varepsilon(x)}_{\frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)} dx = \int_{|x| \ge \rho/\varepsilon} \varphi(x)\, dx$$

By DCT,

$$\lim_{\varepsilon \to 0} \|\chi_\rho \psi_\varepsilon\|_1 = 0.$$

By Hölder's inequality $\left(q = 1 + \frac{q}{p}\right)$ $\to \frac{1}{p} + \frac{1}{q} = 1$!

$$\|\chi_\rho \psi_\varepsilon\|_q = \left(\int_{|x| \ge \rho} |\psi_\varepsilon(x)|^q \, dx \right)^{1/q}$$

$$= \left(\int_{|x| \ge \rho} \underline{\underline{|\psi_\varepsilon(x)|}} \, \underline{\underline{|\psi_\varepsilon(x)|}}^{q/p} \, dx \right)^{1/q}$$

$\psi(x) = \operatorname{ess\,sup}_{|t| \ge |x|} |u(t)|$

$$\cancel{\leq \|\chi_\rho \psi_\varepsilon\|_1^{1/p} \|\chi_\rho \psi_\varepsilon\|_1^{1/q}}$$

$$= \left(\int_{\mathbb{R}^n} \left[(\chi_\rho \psi_\varepsilon)(x) \right] \left[(\chi_\rho \psi_\varepsilon)(x) \right]^{q/r} dx \right)^{1/2q}$$

$$\leq \|\chi_\rho \psi_\varepsilon\|_\infty^{1/p} \|\chi_\rho \psi_\varepsilon\|_1^{1/q} \qquad \varepsilon^n \psi\left(\frac{x}{\varepsilon}\right)$$

$$\|\chi_\rho \psi_\varepsilon\|_\infty = \sup_{|x| \geq \rho} \psi_\varepsilon(x)$$

$$= \sup_{|x| \geq \rho} \rho^{-n} \frac{\rho^n}{\varepsilon^n} \psi_0(\rho/\varepsilon)$$

$$\lim_{\varepsilon\to 0} \|X_1 \psi_2\|_\infty = \rho^n \Big(\lim_{\varepsilon \to 0} \sup_{|x|\geq \rho} \frac{\rho^n}{\varepsilon^n} \psi_0\left(\frac{\rho}{\varepsilon}\right) = 0.$$

$$I_2 \leq A\delta \qquad \forall\ \varepsilon < \varepsilon'.$$

$$= I_1 + I_2 \leq (A+1)\,\delta$$

depend only on ψ.

[If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. $\lim_{\varepsilon \to 0} (f * u_\varepsilon)(x) = f(x) \int_{\mathbb{R}^n} u(t)\,dt.$

whenever x belongs to the Lebesgue set of f].

Corollary: If Φ and its Fourier transform $\hat{\Phi}$ are in $L^1(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$ which is continuous at 0, then

$$f(0) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x) \cdot \Phi(\varepsilon x)\, dx.$$

Proof 0 is a Lebesgue point of f.

Corollary: Suppose $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \geq 0$. If f is continuous at 0, then $\hat{f} \in L^1(\mathbb{R}^n)$ and

$$\boxed{f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, d\xi}$$

for almost every $x \in \mathbb{R}^n$. In particular

$$\boxed{f(0) = \int_{\mathbb{R}^n} \hat{f}(x)\, dx}$$

Proof. Since $\hat{f} \geq 0$, use Fatous lemma, to conclude that

$$f(0) = \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x)\, \Phi(\varepsilon x)\, dx \quad (\Phi \text{ is positive Schwartz function}) \quad (\Phi(0) = 1)$$

$$\geq \int_{\mathbb{R}^n} \liminf_{\varepsilon \to 0} \hat{f}(x)\, \Phi(\varepsilon x)\, dx$$

$$= \int_{\mathbb{R}^n} \hat{f}(x)\, dx \geq 0$$

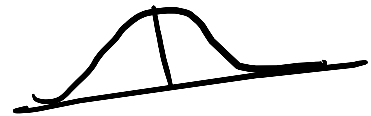
Thm: $\hat{f} \in L^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, \Phi(\varepsilon \xi)\, d\xi$$

$(\Phi(0) = 1$

$\to f$ pointwise a.e.

(as $\varepsilon \to 0$)



$$\sup_{x \in \mathbb{R}^n} \Phi = \|\Phi\|_\infty \le 1$$

$$\to \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, d\xi$$

(by DCT)

pointwise

$$\left[f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle x, \xi \rangle}\, d\xi \right]$$

~~Vx~~ for almost every $x \in \mathbb{R}^n$,

L^2-theory of Fourier transform.

Thm. If $f \in L^1 \cap L^2$, then $\|\hat{f}\|_2 = \|f\|_2$.

Proof: Let $g := \tilde{f}$. $h := f * g \in L^1(\mathbb{R}^n)$

$$\hat{h} = \hat{f} \cdot \hat{g}, \quad \hat{g} = \overline{\hat{f}} \Rightarrow \hat{h} = |\hat{f}|^2 \geq 0$$

$\hat{h} \in L^1(\mathbb{R}^n)$, $h(0) = \int_{\mathbb{R}^n} \hat{h}(x)\,dx$

[h is uniformly continuous because it is the convolution of L^2 functions],

Thus $\|\hat{f}\|_2 = \int_{\mathbb{R}^n} |\hat{f}(x)|^2\,dx = \int_{\mathbb{R}^n} \hat{h}(x)\,dx$

$$= h(0) = \int_{\mathbb{R}^n} f(x)\, g(0-x)\,dx$$

$$= \int_{\mathbb{R}^n} f(x)\,\overline{f(x)}\,dx$$

$$= \|f\|_2.$$

In general, if $f \in L^2(\mathbb{R}^n)$, $\hat{f}$ is the L^2-limit of the sequence of functions defined by,

$$\hat{h_k}(\xi) = \int_{|x| \le k} f(x) \cdot e^{-2\pi i \langle \xi, x \rangle} dx$$

$$= \int_{\mathbb{R}^n} h_k(x) \cdot e^{-2\pi i \langle \xi, x \rangle} dx$$

where $h_k(x) = f(x) \cdot \chi_{\{x \in \mathbb{R}^n ; |x| \le k\}}$,

$h_k \in L^1 \cap L^2$.

Theorem. (Plancherel theorem)

The Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$.

Proof. The Fourier transform is a bounded linear operator defined on the dense subset $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$.

Extend to $L^2(\mathbb{R}^n)$.

$\varphi_k \in \mathcal{S}(\mathbb{R}^n)$

$\varphi_k \to f$ in L^2

$\hat{\varphi}_k \to \hat{f}$ in L^2

Extending definition of Fourier transform to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$

Let $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$

so that $f = f_1 + f_2$, where

$f_1 \in L^1(\mathbb{R}^n)$, $f_2 \in L^2(\mathbb{R}^n)$.

Define, $\hat{f} := \hat{f}_1 + \hat{f}_2$.

$g_1 + g_2 = f_1 + f_2$, $\quad f_1, g_1 \in L^1(\mathbb{R}^n)$, $f_2, g_2 \in L^2(\mathbb{R}^n)$

then, $g_1 - f_1 = f_2 - g_2 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

Since the two definitions of Fourier transform coincide on $L^1 \cap L^2$, we have $\hat{g}_1 - \hat{f}_1 = \hat{f}_2 - \hat{g}_2$

$\Longrightarrow \hat{g}_1 + \hat{g}_2 = \hat{f}_1 + \hat{f}_2$

Note, for $1 \le p \le 2$, $L^p(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

<u>Theorem</u>: $\mathcal{F}: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, has norm 1.

~~$\mathcal{F}: L^{\frac{2}{2-\theta}}(\mathbb{R}^n) \to L^{\frac{2}{1-\theta}}(\mathbb{R}^n)$~~

~~$\mathcal{F}$ is bounded with norm ≤ 1.~~ $0 \leq \theta \leq 1$

$\mathcal{F}: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $\quad p = \frac{2}{1+\theta}$

$1/p = \theta + \frac{1-\theta}{2}$, $\quad \frac{1}{q} = \frac{1-\theta}{2}$

$= \frac{1+\theta}{2}$ $\quad \rightsquigarrow: L^p \to L^q$,

$J: \quad L^{2/(1+\theta)}(\mathbb{R}^n) \longrightarrow L^{2/(1-\theta)}(\mathbb{R}^n).$

$L^{\infty}(\mathbb{R}^n)$

$\delta_0 * 1 \longrightarrow \delta_0.$

Theorem: The inverse of the Fourier transform $\mathcal{F}^{-1}$, can be obtained by letting

$$(\mathcal{F}^{-1} g)(x) = (\mathcal{F} g)(-x) \;,$$

$$\mathcal{F}^{-1} g = \widetilde{\mathcal{F} g} \;, \quad \forall g \in L^2(\mathbb{R}).$$