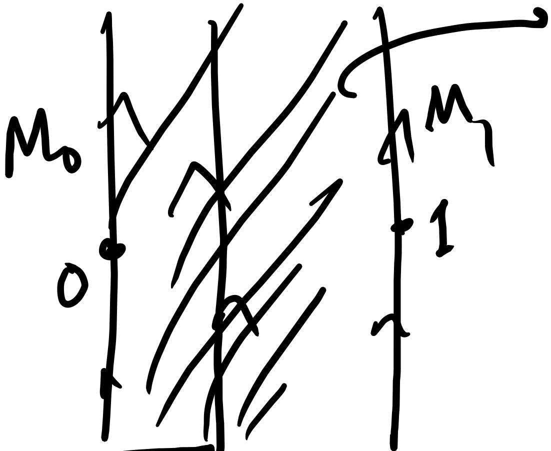
10/08/2021 Lecture - 8

Hadamard three lines lemma



$F \to$ analytic on the open strip.

continuous on the closed strip

bounded on the closed strip

$$\boxed{|F(z)| \le M_0^{1-\theta} \cdot M_1^{\theta}}$$

Re $z = 0$

$0 \le \theta \le 1$

Riesz-Thorin interpolation theorem:

(X, μ), (Y, ν) → measure spaces

S → set of simple functions on X.

T → linear operator defined on S. (to meas. fn. on Y, ν)

For $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, assume that

$$\|T(f)\|_{L^{q_0}(Y,\nu)} \leq M_0 \|f\|_{L^{p_0}(X,\mu)}$$

$$\|T(f)\|_{L^{q_1}(Y,\nu)} \leq M_1 \|f\|_{L^{p_1}(X,\mu)}$$

$f \in S$.

Then for all $0 < \theta < 1$ we have

$$\left[\|T(f)\|_{L^q(Y,\nu)} \le \boxed{M_0^{1-\theta} M_1^{\theta}} \|f\|_{L^p(X,\mu)} \right]$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad f \in S.$$

Proof-.

$$\|Tf\|_{L^q(Y,\nu)} = \sup \left| \boxed{\int (Tf)(x)\, g(x)\, d\nu(x)} \right|$$

where the supremum is taken over all simple functions g on Y with $L^{q'}$-norm less than or equal to 1.

Let, $$g := \sum_{j=1}^{n} \boxed{b_j \, e^{i\beta_j}} \chi_{B_j}$$

where $b_j > 0$, $\beta_j \in \mathbb{R}$, B_j are pairwise disjoint subsets of Y with finite measure.

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \quad (P(0)=1)$$

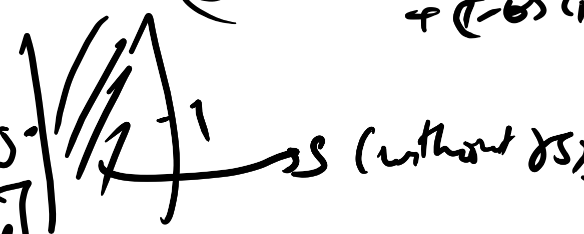
$$Q(z) = \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z \quad (Q(0)=1)$$

$x = \theta a + (1-\theta) b$

$x = \theta(1-a) + (1-\theta)(1-b)$

For $z \in \bar{S} \to$ closed rectangular

confined to a, b



define.

$$F(z) := \int_Y T(f_z)(y) \cdot g_z(y) \, d\nu(y)$$

where; $f_z := \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}$ $\quad (f_0 =$

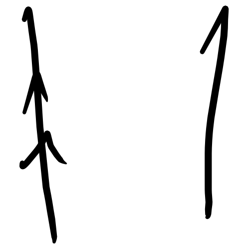
$$g_z := \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j},$$

$$(f_0 \equiv f, \quad g_0 \equiv g).$$

Clearly, $F(z)$ is analytic in z, since $a_k, b_j > 0$

$$\left(F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} \cdot b_j^{Q(z)} \cdot e^{i(\alpha_k+\beta_j)} \underline{\left(\int_Y T(\chi_{A_k})(y) \cdot \chi_{B_j}(y)\, d\nu(y) \right)} \right)$$

Case I : $\operatorname{Re} z = 0$



$$\|f_z\|^{p_0}_{L^{p_0}(X,\mu)} = \int |f_z|^{p_0}\, d\mu$$

$$= \int |f|^{p}\, d\mu \qquad \left(\text{since } |a_n^{P(z)}| = a_n^{p/p_0}\right)$$

and,

$$\|g_z\|^{q_0'}_{L^{q_0'}} = \|g\|^{q'}_{L^{q'}} \qquad \left(\text{since } |b_j^{Q(z)}| = b_j^{q'/q_0'}\right)$$

Case II — $\operatorname{Re} z = 1$

$$\|f_z\|_{L^{p_1}}^{p_1} = \|f\|_{L^p}^{p} \quad \left(\text{since } |a_k^{P(z)}| = a_k^{p/p_1}\right)$$

$$\left(\|g_z\|_{L^{q_1'}}\right)^{q_1'} = \|g\|_{L^{q'}}^{q'} \quad \left(\text{since } |b_j^{Q(z)}| = b_j^{q'/q_1'}\right)$$

Hölder's inequality,

$$|F(z)| \le \|\boxed{T(f_z)}\|_{L^{q_0}} \cdot \|\boxed{g_z}\|_{L^{q_0'}}$$

$$\le \boxed{M_0}\, \|f_z\|_{L^{p_0}} \, \|g_z\|_{L^{q_0'}}$$

$$= M_0 \|f\|_{L^p}^{p/p_0} \cdot \|g\|_{L^{q'}}^{q'/q_0'}$$

(when $\operatorname{Re} z = 0$)

Similarly,

$$|F(z)| \le \boxed{M_1 \|f\|_{L^p}^{p/p_1} \cdot \|g\|_{L^{q'}}^{q'/q_1'}}.$$

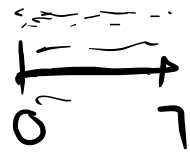
(when $\operatorname{Re} z = 1$).

When $\operatorname{Re} z = \theta$, by Hadamard three line lemma:

$$|F(z)| \le \left(M_0 \|f\|_{L^p}^{p/p_0} \cdot \|g\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1} \cdot \|g\|_{L^{q'}}^{q'/q_1'}\right.$$

$$= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}^{1} \cdot \|g\|_{L^{q'}}^{1}$$

~~Whenever, Re z =~~



$$|F(\theta)| = \left| \int T(f_\theta)(y)\, g_\theta(y)\, d\nu(y) \right| \qquad f_\theta \to f, \quad g_\theta \to g$$

$$\le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}} \qquad (f_\theta = f,\ g_\theta = g)$$

$$\left[\ \|T(f)\|_{L^q(Y,\nu)} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X,\mu)}\ \right]$$

$$\mathcal{F} : \underset{p_0}{L^1(\mathbb{R}^n)} \to \underset{q_0}{L^\infty(\mathbb{R}^n)}.$$ linear operator with bound 1

$$\mathcal{F} : \underset{p_1}{L^2(\mathbb{R}^n)} \to \underset{q_1}{L^2(\mathbb{R}^n)}$$ linear operator with bound 1.

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$$

$$p = \frac{2}{2-\theta}, \quad q = \frac{2}{\theta}$$

$T_\varepsilon : L^{2/(2-\theta)}(\mathbb{R}^n) \to L^{2/\theta}(\mathbb{R}^n)$

$2/(2-\theta)$ takes values in $[1, 2]$.

$$\|M_\varepsilon(f)\|_{L^{1,\infty}} \leq C\, \|f\|_1 \qquad L^1 \to L^{1,\infty}$$

$$\|M_\varepsilon(f)\|_\infty \leq \|f\|_\infty \qquad L^\infty \to L^\infty$$

$$\boxed{L^p \to L^p}$$

Lebesgue differentiation theorem:

$$\boxed{f \overset{??}{=} \sum c_k(f)\, e_k}$$

$e_k = e^{2\pi i k x}$

an "inversion" formula.

L^1-theory of Fourier transform (on Euclidean space).

Def. Let $f \in L^1(\mathbb{R}^n)$. $\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$.

Theorem: (a) The mapping $f \to \hat{f}$ is a bounded linear transformation from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. In fact, $\|\hat{f}\|_\infty \le \|f\|_1$.

(b) If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is ~~$\in C_0$~~ C, is uniformly continuous.

Proof: $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\|\hat{\varphi}\|_\infty \le \|\varphi\|_1$.

$$\varphi_n \longrightarrow f \qquad \|\hat{\varphi}_n\|_\infty \le \|\varphi_n\|_1.$$

$$\hat{\varphi}_n \to g. \qquad \|\widehat{\varphi_n - \varphi_m}\|_\infty \le \|\varphi_n - \varphi_m\|_1,$$

(b) $|\hat{f}(\xi+h) - \hat{f}(\xi)|$

$$\le \int_{\mathbb{R}^n} |f(x)|\,|e^{2\pi i\langle h, x\rangle} - 1|\,dx$$

no dependence on ξ

(Use DCT, to see RHS $\to 0$ as $h \to 0$)

Theorem: (Riemann-Lebesgue) If $f \in L^1(\mathbb{R}^n)$ then $\hat{f}(x) \to 0$ as $|x| \to \infty$ (In view of previous theorem, $\hat{f} \in C_0(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$.

(no simple characterization of image of $L^1(\mathbb{R}^n)$ under $\hat{\ }$ is known)

Theorem: Let $g \in L^1(\mathbb{R}^n)$.

$$L_g : \boxed{L^1(\mathbb{R}^n)} \to \boxed{L^1(\mathbb{R}^n)}$$
$$f \longmapsto g * f$$

$\|f * g\|_1 = \|L_g(f)\|_1 \leq \|g\|_1 \cdot \|f\|_1$ (Young-Hausdorff inequality)

$$\|L_g\| \leq \|g\|_1$$

and, $L_g : \boxed{L^\infty(\mathbb{R}^n)} \to \boxed{L^\infty(\mathbb{R}^n)}$

$$f \longmapsto (f * g)$$

$\|f * g\|_\infty = \|L_g(f)\|_\infty \leq \|g\|_1 \cdot \|f\|_\infty$

$\|L_g\| \le \|g\|_1$

$$L_g : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad (1 \le p \le \infty)$$

$$f \longmapsto \boxed{f * g}$$

$$\boxed{\|L_g(f)\|_p \le \|g\|_1 \|f\|_p}$$

Young-Hausdorff inequality.

Theorem: If $f, g \in L^1(\mathbb{R}^n)$, then

$$\widehat{(f * g)} = \hat{f}\,\hat{g}.$$

(In Fourier series $c_k(f*g) = c_k(f)\, c_k(g)$)

Defn: (Recall) $(\tau_h f)(x) = f(x-h)$ (translation)

$(\delta_a f)(x) = f(ax)$ (dilation)

Note, $(\tau_h f)^\wedge(\xi) = e^{-2\pi i \langle h, \xi\rangle} \hat{f}(\xi)$

$f' = \lim_{h\to 0} \frac{\tau_h f - f}{h}$

$\left(e^{2\pi i \langle x, h\rangle} f(x)\right)^\wedge(\xi) = (\tau_h \hat{f})(\xi)$, for $f \in L^1(\mathbb{R}^n)$

Theorem: If $f \in L^1(\mathbb{R}^n)$, and $\boxed{x_k f(x)} \in L^1(\mathbb{R}^n)$, (where x_k is the k^{th} coordinate function of x.).

Then $\hat{f}$ is differentiable w.r.t. x_k, and

$$\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = \left(\boxed{-2\pi i\, x_k} f(x)\right)^{\wedge}(\xi)$$

Pf: By DCT,

$$\frac{\hat{f}(\xi + h e_k) - \hat{f}(\xi)}{h}$$

$$= \left(\left(\frac{e^{-2\pi i h x_k} - 1}{h}\right) f(x)\right)^{\wedge}(\xi),$$

$\longrightarrow \left(-2\pi i\, x_k\, f(x)\right)^{\wedge}(\xi)$.

$x_k f(x) \in L^1$

$$\boxed{c_k(f) \simeq \frac{1}{k^n}}$$

Note: $p(D) = D^\alpha$.

$$D^\alpha \hat{f}(\xi) = \left((-2\pi i x)^\alpha f(x)\right)^{\wedge}(\xi).$$

$$(D^\alpha f)^{\wedge}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

$p(D)$

Inverting the Fourier transform

Given the Fourier transform on an L^1 function, how do we obtain f from $\hat{f}$?

$$f \overset{??}{=} \sum_{k=-\infty}^{\infty} \underbrace{\boxed{c_k(f)}}_{\text{need not be summable}} e_k$$

$$\boxed{f(x) \overset{??}{=} \underbrace{\int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle \xi, x\rangle}\, d\xi}_{\text{inverse Fourier transform}}}$$

Theorem: (Multiplication Formula)

Let $f, g \in L^1(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} \hat{f}(x) \cdot g(x)\, dx = \int_{\mathbb{R}^n} f(x) \cdot \hat{g}(x)\, dx$$

$$\left[\ \boxed{\hat{u}}(\varphi) := u(\hat{\varphi})\ \right]$$

Pf: Approximate with Schwartz functions...

$\psi_n \to f$ in L^1 $\Rightarrow$ $\boxed{\hat{\psi}_n} \to \boxed{\hat{f}}$ in L^∞

$\varphi_n \to g$ in L^1 $\Rightarrow$ $\boxed{\hat{\varphi}_n} \to \boxed{\hat{g}}$ in L^∞.

Theorem: If $f, \Phi \in L^1(\mathbb{R}^n)$, and $\varphi := \hat{\Phi}$ $\left(\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)\right)$, then

$$\int_{\mathbb{R}^n} \hat{f}(x)\, \boxed{e^{2\pi i \langle x, \xi\rangle}}\, \boxed{\Phi(\varepsilon x)}\, dx$$

$$= \int_{\mathbb{R}^n} f(x)\, \varphi_\varepsilon\left(x \boxed{-\xi}\right) dx, \quad \forall \varepsilon > 0.$$

$$= \left(f * \tilde{\varphi}_\varepsilon\right)(\xi).$$

Pf: Apply multiplication formula to f and $e^{2\pi i \langle \cdot, \xi\rangle}\, \delta_\varepsilon \Phi$

Theorem: (preparatory lemma used in proof of Lebesgue diff-theorem)

Suppose $\varphi \in L^1(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, and for $\varepsilon > 0$, let $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$.

If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, or $f \in C_0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$,

then $\|f * \varphi_\varepsilon - f\|_p \to 0$, as $\varepsilon \to 0$)

Pf:- $\underline{f * \varphi_\varepsilon(x) - f(x)} = \int_{\mathbb{R}^n} (f(x-t) - f(x))\,\varphi_\varepsilon(t)\,dt$

$= \cancel{\int_{\mathbb{R}^n} \ldots}$

$$\tau_n f - f = \tau_n f - \tau_n g + \boxed{\tau_n g - g} + g - f$$

$$\|\tau_n f - f\|_p \leq \cancel{2}\,\|f-g\|_p + \|\tau_n g - g\|_p .$$