07/08/2021

Lecture - 7

$$M_c(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy$$

$\hookrightarrow$ sublinear operator. $M_c(f+g) \le M_c(f) + M_c(g)$.

$M_c : L^1 \longrightarrow L^{1,\infty}$ is bounded.

To Prove : $M_c : L^p \longrightarrow L^p$, is bounded.

(for $1 < p \le \infty$).

$$f_1(x) := \begin{cases} f(x), & \text{if } |f(x)| \geq \alpha/2 \\ 0, & \text{otherwise.} \end{cases} \qquad \Big| \; \alpha > 0$$

$$|f_b(x)| \leq |f_1(x)| + \frac{\alpha}{2}, \quad \forall x \in \mathbb{R}^n.$$

Thus, $$M_{uc}(f)(x) \leq M_{uc}(f_1)(x) + \frac{\alpha}{2}$$

$$\Rightarrow \quad E_{M_{uc}(f),\alpha} \subseteq E_{M_{uc}(f_1),\alpha/2}$$

$$\Rightarrow \quad d_{M_{uc}(f)}(\alpha) \leq \left[d_{M_{uc}(f_1)}(\alpha/2). \right]$$

If $f \in L^p(\mathbb{R}^n)$ $(1 < p \le \infty)$, then $f_1 \in L^1(\mathbb{R}^n)$

$$\|f\|_p^p = p\int_0^\infty t^{p-1} d_f(t)\,dt$$

$$\int_{\mathbb{R}^n} |f|^p\,dx$$

$$\int_{\mathbb{R}^n} |f_1|^p\,dx = \int_{\{x : f(x) > \alpha/2\}} |f|^p\,dx$$

$$\|f_1\|_p^p = \boxed{p\left(\int_{\alpha/2}^\infty \boxed{t^{p-1}} d_f(t)\,dt\right.} + \boxed{\left.\int_0^{\alpha/2} t^{p-1} d_f(t)\,dt\right)}$$

$$\|f_1\|_1 = \int_{\alpha/2}^\infty d_f(t)\,dt + \boxed{\int_0^{\alpha/2} d_f(t)\,dt} \le \boxed{M\int_{\alpha/2}^\infty t^{p-1} d_f(t)\,dt}$$

$$\{x : |f_1(x)| > 0\} = \{x : |f(x)| > \alpha/2\}$$

$$d_{M_{uc}(f)}(\alpha) \leq d_{M_{uc}(f_1)}(\alpha_{/2})$$

$$\leq \boxed{\frac{2A}{\alpha} \|f_1\|_1} \leq \frac{2A}{\alpha}\|f\|_1$$

$g := M_{uc}(f)$ — g

$g := M_{uc}(f_1)$

$$\int_{\mathbb{R}^n} |M_{uc}(f_1)|^p \, dx = p \int_0^\infty \alpha^{p-1} d_{M_{uc}(f)}(\alpha) \, d\alpha$$

$$\|g\|_p^p = p\int_0^\infty \alpha^{p-1}\,\underbrace{\mu\left(\{x : |g|\} E_{g,\alpha}\right)}_{d_g(\alpha)}\,d\alpha$$

$$\leq p\int_0^\infty \alpha^{p-1}\left(\frac{2A}{\alpha}\boxed{\|f_1\|_1}\right)d\alpha$$

$$= p\int_0^\infty \alpha^{p-1}\,\frac{2A}{\alpha}\left(\int_{|f|>\alpha/2} |f(x)|\,dx\right)d\alpha$$

$$= \underline{p(2A)}\int_0^\infty\int_{\mathbb{R}^n} \alpha^{p-2}\,\boxed{|f(x)|\cdot\chi_{\{|f|>\alpha/2\}}}\,dx\,d\alpha$$

$$= p(2A)\int_{\mathbb{R}^n}\left(\int_0^\infty \alpha^{p-2}\,d\alpha\right)dx$$

$|f(x)|\,\underline{\chi_{\{|f|>\alpha/2\}}}$

$$= p(2A) \int_{\mathbb{R}^n} \underline{\underline{\left(\int_0^{2|f(x)|} \alpha^{p-2}\, d\alpha \right)}} |f(x)| \, dx$$

$$= \frac{p(2A)}{p-1} \int_{\mathbb{R}^n} 2^{p-1} |f(x)|^{p-1} |f(x)| \, dx$$

$$= \underline{\underline{\left(\frac{2^p p A}{p-1} \right)}} \cdot \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

$M_{u} : L^p \to L^p$ is bounded.

$M_c : L^p \to L^p$ ~~and~~ is bounded.

For $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,

$M_{HL}(f)$ is finite a.e.

Two preparatory lemmas (before Lebesgue diff. theorem)

Lemma: If $f_n \to f$ in $L^1(\mathbb{R}^n)$, then there exists a subsequence (f_{n_j}) of (f_n) such that

$f_{n_j} \to f$ pointwise a.e.

Pf: $N_1 \le N_2 \le \cdots$

$\|f_m - f_n\| < \frac{1}{2^k}$, whenever $m, n \geq N_k$.

$$f_{N_m} = f_{N_1} + \sum_{l=1}^{m-1} \left(f_{N_{l+1}} - f_{N_l}\right)$$

converges pointwise (as $m \to \infty$) a.e. because:

$$g_m := |f_{N_1}| + \sum_{l=1}^{m-1} |f_{N_{l+1}} - f_{N_l}|$$

converges pointwise (as $m \to \infty$) a.e.

By monotone convergence theorem,

$$g \in \mathcal{L}^1(\mathbb{R}^n).$$

$$\Rightarrow \boxed{g} < \infty \quad \text{a.e.}$$

$$f_{N_m} \longrightarrow \boxed{\tilde{f}} \quad \text{pointwise a.e.}$$

$$\tilde{f} \in \mathcal{L}^1(\mathbb{R}^n).$$

$$0 \leq \int |f - \tilde{f}|\,dx$$

$$= \int \boxed{|\lim_{m\to\infty} f_{N_m} - f|}\,dx \leq \lim_{m\to\infty} \int |f_{N_m} - f|\,dx$$

Thus, $f_{N_k} \to \tilde{f}$ in $L^1(\mathbb{R}^n)$.

$\Rightarrow$ $\boxed{f = \tilde{f} \text{ a.e.}}$

f_{N_k} converges pointwise to f.

<u>Lemma</u>: $\varphi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$.

$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)$. If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$

then $\| f * \varphi_\varepsilon - f \|_p \longrightarrow 0$ as $\varepsilon \to 0$

Pf: $(f * \varphi_\varepsilon)(x) - f(x)$

$$= \int_{\mathbb{R}^n} \boxed{[f(x-t) - f(x)]} \, \varphi_\varepsilon(t) \, dt$$

By Minkowski's inequality for integrals,

$$\|f * \varphi_\varepsilon - f\|_p \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-t) - f(x)|^p \, dx \right)^{1/p} \varphi_\varepsilon(t) \, dt$$

$$\|\tau_t f - f\|_p \le \boxed{\|\tau_t f - \tau_t g\|_p} + \boxed{\|\tau_t g - g\|_p}$$

$= \|g - f\|_p$

(Assume $g \in C_c(\mathbb{R}^n)$)

$$\| f * \varphi_\varepsilon - f \|_p$$

$$\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-\varepsilon t) - f(x)|^p \, dx \right)^{1/p} \varphi(t)\, dt$$

Use DCT,

$$\lim_{\varepsilon \to 0} \underbrace{\left(\int_{\mathbb{R}^n} |f(x-\varepsilon t) - f(x)|^p \, dx \right)^{1/p}}_{g(\varepsilon t)} = 0 \quad \text{for every } t \in \mathbb{R}^n$$

$$\boxed{\lim_{\varepsilon \to 0} \| f * \varphi_\varepsilon - f \|_p = 0}$$

Proof of Lebesgue differentiation theorem;

$$f_r(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

$$= \frac{1}{|B(0,r)|}\int_{B(0,r)} f(y-x)\,dy$$

$$= \frac{1}{|B(0,r)|}\int_{\mathbb{R}^n} f(y-x)\boxed{\chi_{B(0,r)}}\,dy = f * \frac{\chi_{B(0,r)}}{|B(0,r)|}$$

$f_r \longrightarrow f$ in $L^1(\mathbb{R}^n)$.

To Show : $\boxed{\lim_{r \to 0} f_r(x) \text{ exists a.e.}}$

Pf. For $g \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\underline{\underline{\Omega g(x)}} = \left| \limsup_{r \to 0} g_r(x) - \liminf_{r \to 0} g_r(x) \right|$$

represents oscillation of the family $\{g_r\}$ as $r \to 0$

If $g \in C_c(\mathbb{R}^n)$, then $g_r \to g$ uniformly,

and thus $\underline{\underline{\Omega g \equiv 0}}$.

$$d_{2M_c(g),\varepsilon} \le \frac{2A}{\varepsilon}\underline{\|g\|}$$

Clearly,

$$\underline{\underline{\Omega g}}(x) \le \boxed{2M_c\, g(x)}$$

If $f = h + \boxed{g}$, where $\underline{h} \in C_c(\mathbb{R}^n)$

and $g \in L^1(\mathbb{R}^n)$, then

$$\Omega f \leq \Omega h + \Omega g.$$

$$\limsup_{r \to 0} (h_r + g_r) \leq \limsup_{r \to 0} h_r + \limsup_{r \to 0} g_r$$

$$-\liminf_{r \to 0} (h_r + g_r) \leq -\left(\liminf_{r \to 0} h_r + \liminf_{r \to 0} g_r\right)$$

Since $h \in C_c(\mathbb{R}^n)$

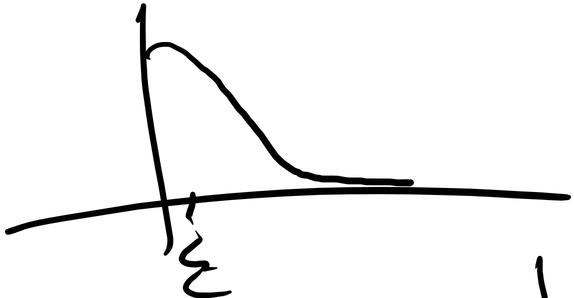
$$\boxed{\Omega f \le \Omega g}$$

$$\underline{\underline{d_{\Omega f, \varepsilon}}} \le d_{\Omega g, \varepsilon} \le d_{2M_2 g, \varepsilon}$$

$$\le \frac{2A}{\boxed{\varepsilon}} \underline{\underline{\|g\|_1}}$$

$$\Rightarrow \quad d_{\Omega f, \varepsilon} \le 0, \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad d_{\Omega f, \varepsilon} = 0, \qquad \forall \varepsilon > 0$$



$\Rightarrow \quad \Omega f = 0 \quad$ a.e.

$\Rightarrow \lim_{r\to 0} f_r$ exists a.e. (and hence $\lim_{r\to 0} f_r(x) = f(x)$

for almost every $x \in \mathbb{R}^n$.

Lebesgue set of f $\in L^1_{loc}(\mathbb{R}^n)$.

$$\left\{ x \in \mathbb{R}^n : \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-t) - f(x)|\, dt \to 0 \text{ as } r \to 0 \right\}.$$

$$\left| \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x-t) - f(x)\, dt \right| \leq$$

Prop^n: Lebesgue set of $f \in L^1_{loc}(\mathbb{R}^n)$ includes almost every point of $\mathbb{R}^n$.

Proof: Let $\rho \in \mathbb{Q}$.

$|f(t) - \rho| \in L^1_{loc}(\mathbb{R}^n)$

F_ρ be the complement of the set of all $x \in \mathbb{R}^n$ where

$$\lim_{r \to 0} \frac{1}{B(0,r)} \int_{B(0,r)} \big(|f(x-t) - \rho| - |f(x) - \rho| \big)\, dt = 0$$

By Lebesgue diff. theorem,

$\mu(F_p) = 0$

Pick $x \in \mathbb{R}^n - \bigcup_{p \in \mathbb{Q}} F_p$

Claim: $x \in$ Lebesgue set of f.

Pf. Let $p \in \mathbb{Q}$ be such that $|f(x) - p| < \frac{\varepsilon}{2}$.

$$\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-t) - f(x)|\, dt$$

$$\leq \frac{1}{|B(0,r)|} \left(\int_{B(0,r)} |f(x-t) - p|\, dt + \int_{B(0,r)} |p - f(x)|\, dt \right)$$

$$\lim_{r\to 0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-t) - f(x)|\, dt$$

$$\leq \lim_{r\to 0} \frac{1}{|B(0,r)|} \left(\int_{B(0,r)} \boxed{|f(x-t) - \rho|}\, dt \right)$$

$$+ \lim_{r\to 0} \frac{1}{|B(0,r)|} \left(\int_{B(0,r)} |f(x) - \rho|\, dt \right)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall \varepsilon > 0.$$

$$\Rightarrow \lim_{r\to 0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-t) - f(x)|\, dt = 0$$

where $x \in \left(\bigcup_{\rho \in \mathbb{Q}} F_\rho \right)^c$.

~~Since~~ $\lim_{r\to 0} f_r(x) = f(x)$ a.e.;

$\lim_{r\to 0}$ avg. of f on ball of radius r

$f = g + h$

Theorem (Riesz Thorin interpolation theorem)

Let (X,μ), and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X, and taking values in the set of measurable functions over Y.

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that,

$$\| \underline{T(f)} \|_{L^{q_0}(Y,\nu)} \leq M_0 \|f\|_{L^{p_0}}$$

$$\| \underline{T(f)} \|_{L^{q_1}(Y,\nu)} \leq M_1 \|f\|_{L^{p_1}}$$

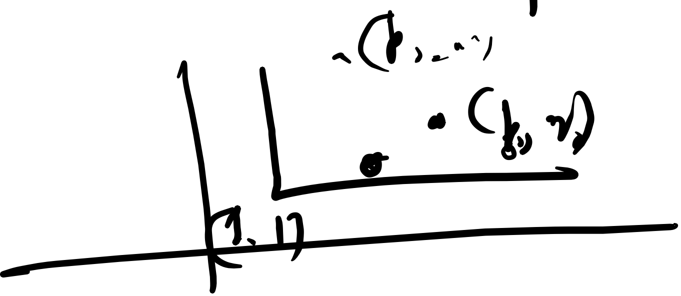
for all simple functions f on X.

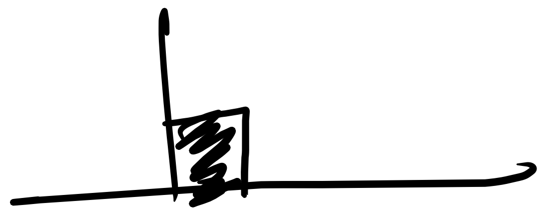
Then for all $0 < \theta < 1$, we have

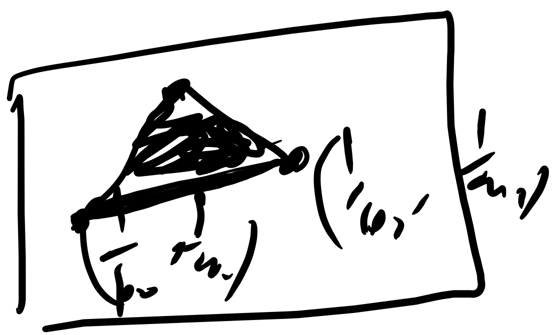
$$\|T(f)\|_{L^q} \leq \boxed{M_0^{1-\theta} M_1^{\theta}} \|f\|_{L^p}.$$

for all simple function f on X where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$







$$\mathcal{F}: L^{\overset{p_0}{1}} \to L^{\overset{q_0}{\infty}}, \quad \|\mathcal{F}(f)\|_{L^\infty} \le \|f\|_{L^1}$$

$$\mathcal{F}: L^{\overset{p_1}{2}} \to L^{\overset{q_1}{2}}, \quad \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}.$$

$$\frac{1}{p} = 1-\theta + \frac{\theta}{2} = \frac{1+\theta}{2} \Rightarrow p = \frac{2}{1+\theta}$$

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2} \Rightarrow q = \frac{2}{\theta}$$

$$\mathcal{F}: L^{4/3}(\mathbb{R}^n) \to L^4(\mathbb{R}^n) \qquad \boxed{0 \le \theta \le 1}$$

For $1 \le p \le 2$, $f \in L^p(\mathbb{R}^n)$

$\mathcal{F}(f) \in L^1_{loc}(\mathbb{R}^n)$.

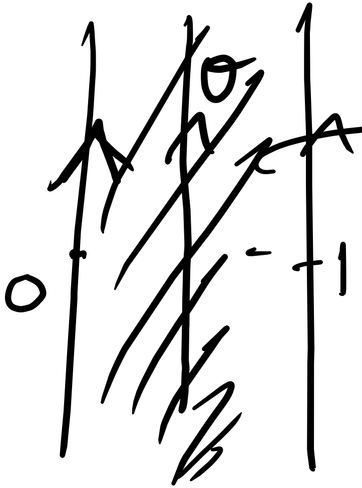
$p = \infty$ $f \equiv 1$.

$\mathcal{F}(1)$ $\boxed{\mathcal{F}\cancel{(f)} = \boxed{\delta_0}}$

(For $\|M_{uc}(f)\|_{L^{1,\infty}} \le C \|f\|_{L^1}$ $\quad \|M_{uc}(f)\|_p \overset{??}{\lesssim} C\|f\|_p$

$\|M_{uc}(f)\|_{L^\infty} \le \|f\|_{L^\infty}$)

Preparatory lemma: (Hadamard's three lines lemma)



$S = \{ z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \}$

Let F be analytic on S and continuous and bounded on the closure, such that

$\boxed{|F(z)| \leq B_0}$ when $\operatorname{Re} z = 0$.

$|F(z)| \leq B_1$, when $\operatorname{Re} z = 1$.

$0 < B_0, B_1 < \infty$

Then, $|F(z)| \leq B_0^{1-\theta} B_1^{\theta}$

where $\operatorname{Re} z = \theta$, for any $0 \leq \theta \leq 1$

Pf: $\boxed{G(z) := F(z)\left(B_0^{1-z} B_1^{z}\right)^{-1}}$

F is bounded on the closed unit strip.

and $B_0^{1-z} B_1^{z}$ is bounded from below.

$$\left| B_0^{1-z} B_1^{z} \right| = B_0^{1-x} B_1^{x} > 0. \qquad (z = x + iy)$$

Thus $|G(z)| \le M, \ \forall z \in S$

(for some $M > 0$).

On ∂S,

$|G(z)| \le 1, \ \forall z \in \partial S$.

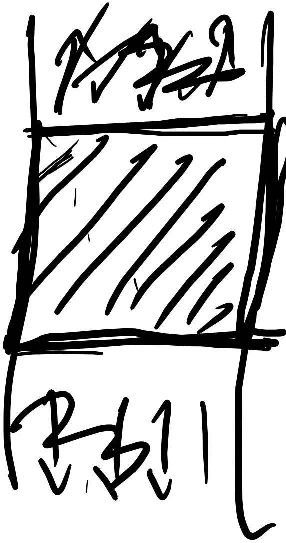
($z = x + iy$)

$$|G(z) e^{z^2 - 1/n}| \le M e^{-y^2/n} e^{x^2 - 1/n} \le M e^{-y^2/n}$$

G_n

$G_n(x+iy) \to 0$ uniformly, on $0 \le x \le 1$

as $|y| \to \infty$.



$|G_n(x+iy)| \le 1$ with

By maximum modulus princ.?

$$|G_n(z)| \le 1$$

$$\Rightarrow |G_n(z)| \le 1 \quad \forall z \in S.$$

$$|G(z)| = \lim_{n\to\infty} |G_n(z)| \le 1 \quad \forall z \in S$$

$$|F(z)| \le B_0^{1-\operatorname{Re} z} \cdot B_1^{\operatorname{Re} z},$$

$$= B_0^{1-\theta} B_1^{\theta},$$