06/08/2021 Lecture - 6

Previous lecture. $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

is a homeomorphism.

$\rho_{\alpha\beta}(f) = \sup_{x \in \mathbb{R}^n} \| x^\alpha \partial^\beta f \| < \infty$. $\frac{C}{(1+|x|)^N}$

$\sigma_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} \| \partial^\alpha (x^\beta f) \| < \infty$.

$\boxed{\| \hat{f} \|_\infty \leq \| f \|_{L^1}}$

Today's lecture.

Lebesgue differentiation theorem.

Standard theory of calculus

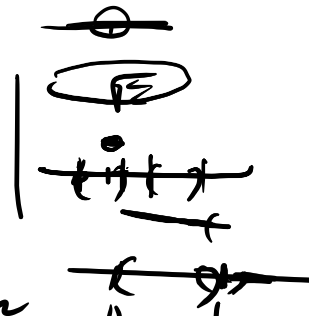
— "nice" subsets of $\mathbb{R}^n$

"nice" function on those "nice" subsets.

In measure theory.

"measurable" subsets of $\mathbb{R}^n$

"measurable" functions over $\mathbb{R}^n$.



Main features:

(1) differentiation of the integral.

(2) covering lemma: splitting the space.

(3) behavior near a 'general' point of an arbitrary set. (almost everywhere pointwise behavior).

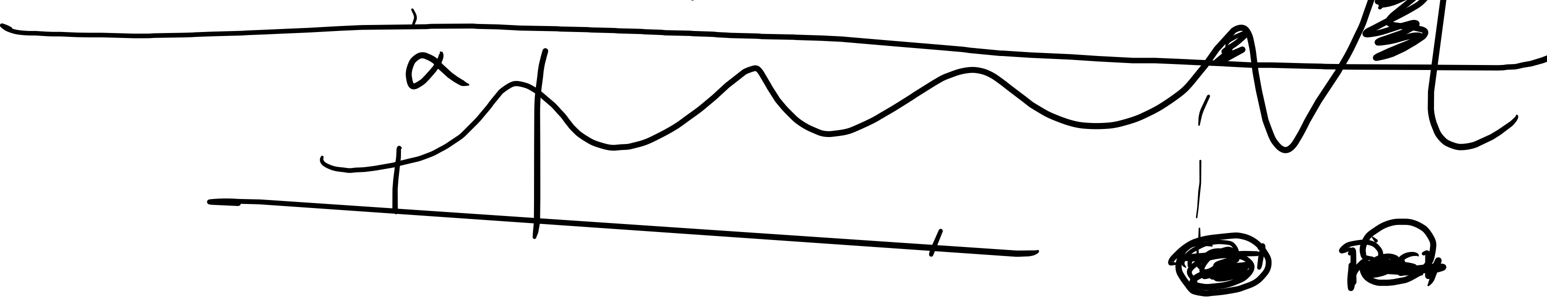
(4) splitting a function into their large and small parts.

Theorem: (Fundamental theorem of Lebesgue, FTOC, Lebesgue differentiation theorem):

If $f \in L^1_{loc}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$ (w.r.t. Lebesgue measure), we have,

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(y)\,dy = f(x).$$

centre radius



If $f \in L^1(\mathbb{R}^n)$,

$E_\alpha = \{ x \in \mathbb{R}^n : |f(x)| > \alpha \}$

(Chebyshev's inequality)

$$\alpha \mu(E_\alpha) \leq \int_{E_\alpha} |f(x)|\,dx \leq \|f\|_1$$

$$\Rightarrow \boxed{\mu(E_\alpha) \leq \frac{1}{\alpha} \|f\|_1}$$

Distribution function of f on $\mathbb{R}^n$ (f measurable)

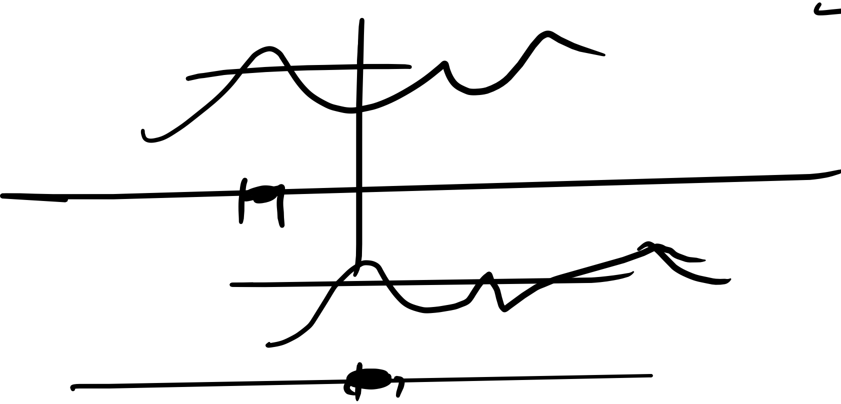
Def^n - Let (X, μ) be a measure space and f be a measurable function on X. The distribution function of f, d_f, is defined on $[0, \infty)$ as follows:

$$d_f(\alpha) := \mu\left(\underline{\{x \in X ; |f(x)| > \alpha\}}\right).$$

It gives information about the 'size' of f, but not about behaviour of f itself at any point of X.

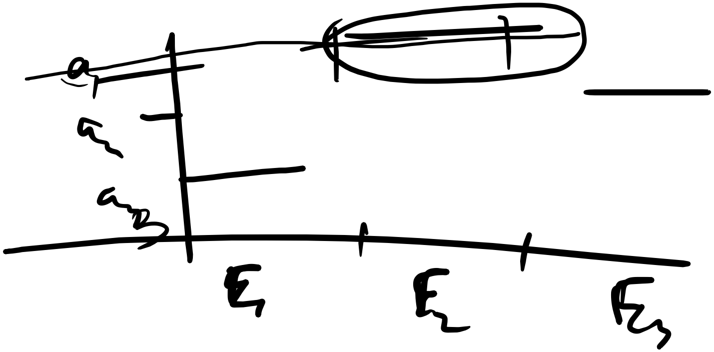
Let $X = \mathbb{R}^n$, $\mu \to$ Lebesgue measure.

f and all of its translates have the same distribution function.

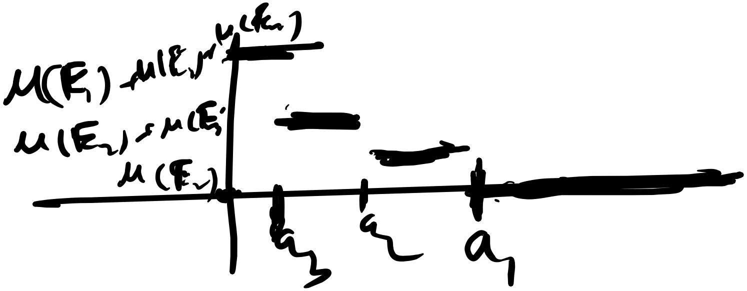


$$\boxed{d_f : [0, \infty) \to \mathbb{R}_+ \cup \{\infty\}}$$

d_f is a decreasing function.
(not necessarily strictly)



$$f = a_3 \chi_{E_1} + a_2 \chi_{E_2} + a_1 \chi_{E_3}$$



$$x \geq a_1$$

$$E_a = \phi$$

E_{a_3}, E_{a_2}

Proposition: Let f, g be measurable functions on (X, μ). Then $\forall\, \alpha, \beta > 0$ we have:

(1) If $|g| \le |f|$, μ-a.e., then $d_g \le d_f$.

(2) $d_{cf}(\alpha) = d_f\left(\frac{\alpha}{|c|}\right)$, $\forall\, c \in \mathbb{C} \setminus \{0\}$

(3) $d_{f+g}(\alpha+\beta) \le d_f(\alpha) + d_g(\beta)$,

(4) $d_{fg}(\alpha\beta) \le d_f(\alpha) + d_g(\beta)$.

Pf: $E_{g,\alpha} \subseteq E_{f,\alpha} \Rightarrow \mu(E_{g,\alpha}) \le \mu(E_{f,\alpha})$

$$\boxed{E_{f+g,\,\alpha+\beta} \subseteq \underline{E_{f,\alpha}} \cup E_{g,\beta}}$$

$$\{x \in \mathbb{R}^n : |f(x)+g(x)| > \alpha+\beta\}$$

$$\boxed{|f(x)| +} |g(x)| > \alpha+\beta$$

* Knowledge of d_f provides sufficient information to compute $\boxed{\|f\|_p}$.

Proposition. For $f \in L^p(X, \mu)$, $0 < p < \infty$, we have $\left(\|f\|_p^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha\right)$.

Rmk: If $f \in L^\infty(X, \mu)$, then $\|f\|_\infty = \inf\{\alpha : d_f(\alpha) = 0\}$

Pf: $p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha$

$$= p \int_0^\infty \alpha^{p-1} \left(\int_{\boxed{\{x : |f(x)| > \alpha\}} \to X} \boxed{\chi_{\{x : |f(x)| > \alpha\}}}\, d\mu\right) \cdot d\alpha$$

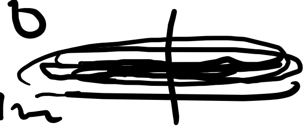
$$= \int_X \left(\int_0^{|f(x)|} p\,\alpha^{p-1}\, d\alpha \right) d\mu(x)$$

$$= \int_X |f(x)|^p \, d\mu(x) = \|f\|_p^p .$$

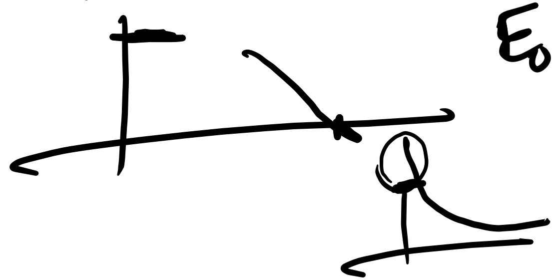
Fubini's theorem.

* The (rate of) decrease of $d_f(\alpha)$ as α grows describes the relative largeness of f (of local importance)

The (rate of) increase of α as $\alpha \to 0^+$



describes relative smallness of the function (of global importance) but is of no interest, if for example the function is supported on a bounded set.



$E_0 := \{x \in \mathbb{R}^n : |f(x)| > 0\}$

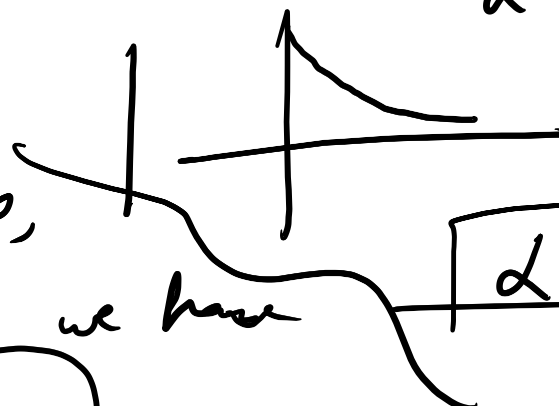
Weak $L^p(X,\mu)$:

$0 < p < \infty$.

$$\|f\|_{L^{p,\infty}} := \inf\left\{ C > 0 : d_f(\alpha) \le \frac{C^p}{\alpha^p}, \forall \alpha > 0 \right\}$$

$< \infty$

$\Rightarrow f \in$ weak $L^p(X,\mu)$



$\alpha^p d_f(\alpha) \le C^p$

Prop^n. For any $0 < p < \infty$, and any $f \in L^p(X,\mu)$, we have

$$\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}.$$

Hence $L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu)$

Pf: $\alpha^p \, d_f(\alpha) \le \int_{E_{f,\alpha}} |f(x)|^p \, dx.$

$$\le \|f\|_p^p \rightsquigarrow C^p$$

* The ~~[illegible]~~ inclusion $L^p \subseteq L^{p,\infty}$ is strict, the same about $L^{1,\infty}$

(Lorentz spaces: $L^{p,q}(X,\mu)$, $q \in (0,\infty)$. $L^{p,p}(X,\mu) = L^p(X,\mu)$)

$$\left[\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy \overset{?}{=} \underline{\underline{f(x)}} \right]$$

↳ We want to understand this.

$$\lim_{r} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} \big(f(x) - f(y)\big)\,dy \right| \to 0$$

Consider the quantitative analogue...

$$M_{\underline{c}}[f](x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy.$$

for f measurable $(\mathbb{R}^d, \mu)$.

M_c — maximal function of f.

Hardy-Littlewood maximal function

(centred) ~~Hardy-Li~~ HL maximal function

Def^n. $$M_{uc}(f)(x) = \sup_{\substack{r>0 \\ y \in B(x,r)}} \underbrace{\frac{1}{|B(y,r)|} \int_{B(y,r)} |f(z)|\, d\mu(z)}_{\text{avg}(f)}$$

(supremum of the average of $|f|$, over all open balls $B(y,r)$ that contain x.

Theorem: Let f be a measurable function defined on $\mathbb{R}^n$.

(a) If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then $M_c(f)$ is finite a.e.

(b) If $\boxed{f \in L^1(\mathbb{R}^n)}$, then $M_c(f) \in \boxed{L^{1,\infty}(\mathbb{R}^n)}$

(weak $L^1(\mathbb{R}^n)$)

i.e. For every $\alpha > 0$,

$$\mu(\{x \in \mathbb{R}^n : M_c(f)(x) > \alpha\}) \le \frac{\boxed{3^n}}{\alpha} \int_{\mathbb{R}^n} |f| \, d\mu(x)$$

$$\le \frac{3^n}{\alpha} \|f\|_1$$

(C) If $f \in L^p(\mathbb{R}^n)$, with $1 < p \leq \infty$,
then $M_\varepsilon f \in L^p(\mathbb{R}^n)$, and

$$\|M_\varepsilon f\|_p \leq A_{p,n} \|f\|_p .$$

$$\boxed{\frac{3^n}{p(p-1)}}$$

$$\boxed{\frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f(y)\,dy}$$

$\frac{1}{B_{\varepsilon,\cdot}}$

$B_{\varepsilon,\cdots}$

$f * \varphi_\varepsilon \to f$ a.e.

$$\boxed{\|f * \varphi_\varepsilon - f\|_1 \to 0}$$

First observation:

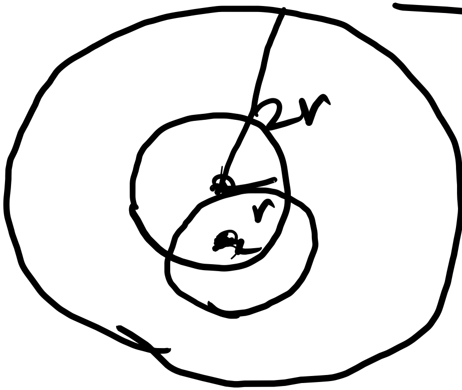
$M_c(f)(x)$ $\quad M_{uc}(f)(x)$

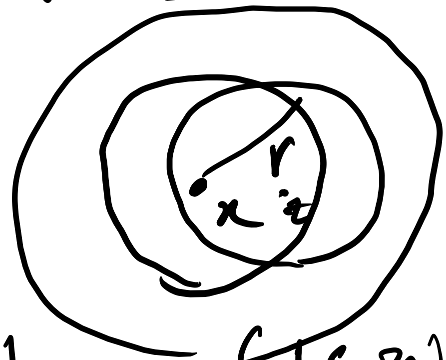
$M_c(f)(x)$: sup. of averages of f over balls centered at x.

$M_{uc}(f)(x)$: sup of averages of f over balls containing x

$$\boxed{M_c(f)(x) \leq M_{uc}(f)(x)} \qquad \forall x \in \mathbb{R}^n.$$

$$M_{uc}(f)(x) \le \boxed{2^n} M_c(f)(x), \quad \forall x \in \mathbb{R}^n$$





$y \in B(x,r)$

$$\frac{1}{|B(x,r)|}\int_{B(y,r)} |f(z)|\,d\mu(z) \le \frac{2^n}{|B(x,2r)|}\int_{B(x,2r)} |f(z)|\,d\mu(z) = 2^n M_c(f)(x)$$

Lemma: (Covering lemma) - Vitali's

Let $\{B_1, \ldots, B_K\}$ be a finite collection of open balls in $\mathbb{R}^n$. $\exists$ finite subcollection of pairwise disjoint balls such that:

$$\sum_{r=1}^{l} \boxed{|B_{j_r}|} \geq \boxed{3^{-n}} \left| \bigcup_{i=1}^{K} B_i \right|.$$

$$\left| \int_{B_i} f \; f \ldots \right]$$

Pf. $|B_1| \geq |B_2| \geq \cdots \geq |B_n|$.

$\boxed{j_1 = 1}$

j_2 ← smallest index > 1 s.t.

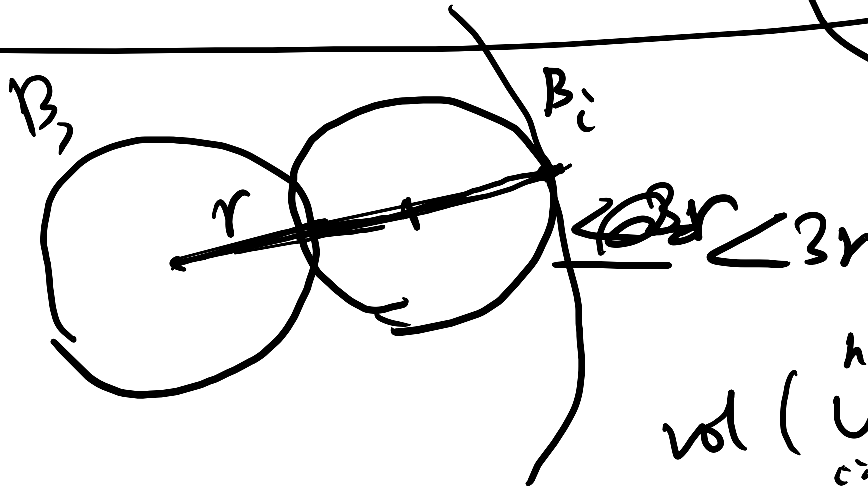
$$B_{j_1} \cap B_{j_2} = \phi.$$

Having chosen $j_1 \ldots j_l$, let j_{l+1} be the least index $s > j_l$ s.t.

$\boxed{\bigcup_{m=1}^{l} B_{j_m}}$

B_s is disjoint from B_{j_l}.

$$\boxed{\bigcup_{i=1}^{k} B_i \;\boxed{\subseteq}\; \bigcup_{\gamma=1}^{l} \boxed{3 B_{j_\gamma}}}$$



centre is the same, radius has become 3-fold.

$$\mathrm{vol}\left(\bigcup_{i=1}^{n} B_i\right) \le 3^n \sum_{\gamma=1}^{l} |B_{j_\gamma}|$$

Pf:

(a) Since $M_{uc}(f) \geq M_c(f)$, we have

$$\{x \in \mathbb{R}^n : M_c(f)(x) > \alpha\}$$

$$\subseteq \{x \in \mathbb{R}^n : |M_{uc}(f)(x)| > \alpha\}$$

$$\boxed{E_{M_c(f),\alpha} \subseteq E_{M_{uc}(f),\alpha}.}$$

Claim: $E_{M_{uc}(f),\alpha}$ is an open set.

Proof of Claim: $x \in E_{M_{uc}(f),\alpha}$.

$\exists$ open ball B_x containing x such that avg. of $|f|$ over B_x is strictly bigger than α. For $y \in B_x$, $M_{\boxed{uc}}(f)(y)$ is also bigger than α.

$\Rightarrow$ $y \in E_{M_{uc}(f),\alpha}$ $\Rightarrow$ $B_x \subseteq E_{M_{uc}(f),\alpha}$

$K \to$ compact subset of E_α.

For each $x \in K$, $\exists$ an open ball B_x

containing x such that

$$\frac{1}{|B_x|} \int_{B_x} |f(y)|\, dy > \alpha.$$

$\exists$ a finite subcover $\{B_{x_1}, \ldots, B_{x_k}\}$ of K

$$|K| \leq \left|\bigcup_{i=1}^{k} B_{x_i}\right| \leq 3^n \sum_{i=1}^{l} |B_{x_{y_i}}|$$

$$\leq \frac{3^n}{\alpha} \left(\sum_{i=1}^{l} \int_{B_{x_{y_i}}} |f(y)|\, dy \right)$$

$$\le \frac{3^n}{\alpha} \int_{E_{M_{HL}(f),\alpha}} |f(y)|\, dy.$$

Using inner-regularity of Lebesgue measure.

$$\underline{\underline{d_{M_{HL}(f)}(\alpha)}} \le \frac{3^n \|f\|_1}{\alpha}$$

$M_{HL} : L^1 \longrightarrow L^{1,\infty}$ is bounded

$M_c : L^1 \longrightarrow L^{1,\infty}$ is bounded.

$M_m : L^\infty \to L^\infty$ is bounded.

$\Rightarrow$ $\boxed{M_1 : L^\infty \to L^\infty \text{ is bounded!}}$