03/08/2021 — Lecture - 5

Def^n: $f \in \mathcal{S}(\mathbb{R}^n)$. Define.

inner product on $\mathbb{R}^n$

$$\left[\ \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx\ \right]$$

$\hat{f} \to$ Fourier ~~trans~~ transform of f.

dim of $\mathbb{R}^n$

$$\|f\|_\infty \le \frac{C}{(1+|x|)^{\boxed{n+1}}}$$

$$\sqrt{x_1^2 + \dots + x_n^2}$$

$$\int |f|\, dx \le C \int \frac{1}{(1+|x|)^{n+1}}\, dx$$

Defn: (1) $\tau_h : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $h \in \mathbb{R}^n$,

$$(\tau_h f)(x) = f(x-h)$$ (translation)

(2) $$(\delta_a f)(x) = f(ax)$$ (dilation by a)

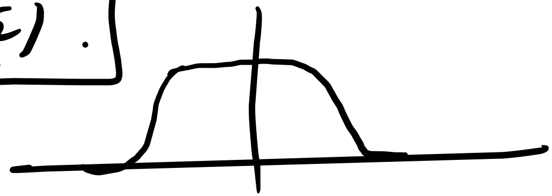
$\delta_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ typically $a > 0$

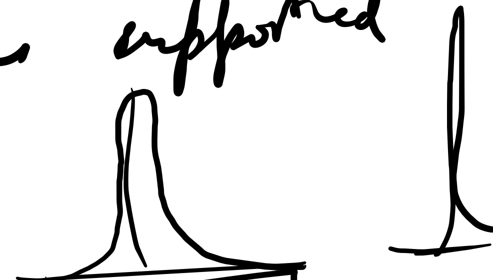
(3) $\tilde{\ } : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

$$\tilde{f}(x) = f(-x),$$ (reflection)

(4) For $f \in \mathcal{S}(\mathbb{R}^n)$, $$\boxed{f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)}$$ (for $a > 0$)

If f is supported on the unit ball $B(0,1)$ of $\mathbb{R}^n$, then f_ε is supported on $\boxed{B(0,\varepsilon)}$.





$\boxed{f * \varphi_{\frac{1}{n}}} \to f$

Proposition: Let $f, g \in S(\mathbb{R}^n)$.

(1) $\|\hat{f}\|_\infty \leq \|f\|_1$.

(2) $\widehat{f+g} = \hat{f} + \hat{g}$

(3) $\widehat{bf} = b\hat{f}$

(4) $\hat{\hat{f}} = \hat{\tilde{f}}$

(5) $\hat{\bar{f}} = \overline{\tilde{\hat{f}}}$

Schwartz topology is finer than the L^1-norm topology

Pf: (1) $\mathcal{F}: (\mathcal{S}(\mathbb{R}^n), \|\cdot\|_1) \longrightarrow (\mathcal{S}(\mathbb{R}^n), \|\cdot\|_\infty)$.

subspace of $L^1(\mathbb{R}^n)$. subspace of $L^\infty(\mathbb{R}^n)$.

$$|\mathcal{F}(f)(\xi)| = \left|\left(\int_{\mathbb{R}^n} f(x) \boxed{e^{-2\pi i\langle x, \xi\rangle}}\, dx\right)\right|$$

$$\leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_1$$

$\Rightarrow \|\mathcal{F}(f)\|_\infty \leq \|f\|_1 \ldots$

(4) $$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\langle x,\xi\rangle}\, dx$$

$$\mathcal{F}(\tilde{f})(\xi) = \int_{\mathbb{R}^n} f(-x)\, e^{-2\pi i\langle x,\xi\rangle}\, dx$$

$$= \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\langle x,-\xi\rangle}\, dx$$

$$= \mathcal{F}(f)(-\xi) = \widetilde{\mathcal{F}f}(\xi)$$

$$\hat{\tilde{f}} = \tilde{\hat{f}}.$$

(5) $$\mathcal{F}(\bar{f})(\xi) = \overline{\int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\langle x,-\xi\rangle}\, dx}$$

$$= \overline{\mathcal{F}(f)(-\xi)} = \overline{\tilde{\hat{f}}}$$

Proposition: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$, $t > 0$.

$$|e^{-2\pi i \langle y, \xi \rangle}| = 1$$

(1) $\widehat{\tau_y f}(\xi) = \underbrace{\boxed{e^{-2\pi i \langle y, \xi \rangle}}}_{\text{modulation}} \hat{f}(\xi)$

τ_y: translation

(2) $\big(\underbrace{e^{2\pi i \langle x, y \rangle}}_{\text{modulation}} f(x)\big)^\wedge(\xi) = \big(\underbrace{\tau_y}_{\text{translation}}(\hat{f})\big)(\xi)$

(3) $\delta_t(f)^\wedge = t^{-n} \delta_{1/t}(\hat{f}) = (\hat{f})_t$

Pf. (1) $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$

$$\widehat{(\tau_y f)}(\xi) = \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i \langle x-y, \xi \rangle} e^{-2\pi i \langle y, \xi \rangle} dx$$

$$= e^{-2\pi i \langle y, \xi \rangle} \left(\int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx \right)$$

$$= e^{-2\pi i \langle y, \xi \rangle} \hat{f}(\xi).$$

(2) $\left(e^{2\pi i \langle x, y \rangle} f \right)^{\wedge}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i \langle x, y \rangle} f(x) \cdot e^{-2\pi i \langle x, \xi \rangle} dx$

$$= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi - y \rangle} dx$$

$$= \hat{f}(\xi - y)$$

$$= \tau_y(\hat{f})(\xi).$$

(3) $\delta_t(f)^\wedge(\xi) = \int_{\mathbb{R}^n} f(tx)\, e^{-2\pi i \langle x, \xi \rangle} dx$

$(t > 0)$

$$= \frac{1}{t^n} \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle \frac{x}{t}, \xi \rangle} dx$$

$$= \frac{1}{t^n} \hat{f}\left(\frac{\xi}{t}\right) = (\hat{f})_t(\xi)$$

Proposition. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{Z}_+^n$.

$$\boxed{\frac{d}{dx_i} f = \lim_{h \to 0} \frac{\tau_h f - f}{h}}$$

(1) $(\partial^\alpha f)^\wedge(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

differentiation → multiplication

$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$

(2) $\left((-2\pi i x)^\alpha f(x)\right)^\wedge(\xi) = (\partial^\alpha \hat{f})(\xi)$.

multiplication → differentiation.

(3) $\boxed{\hat{f} \in \mathcal{S}(\mathbb{R}^n)}$.

$O: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

$f \to f(O\cdot)$

(4) $\widehat{f * g} = \hat{f} \cdot \hat{g}$, (5) $\widehat{(f \circ O)}(\xi) = \hat{f}(O\xi)$ $\boxed{f \circ O = O f}$

O: orthogonal matrix

ξ – column vector.

Pf. (1) $(\partial^\alpha f)^\wedge(\xi)$

$|\alpha|=1$

$(0,\dots,0,1,0,\dots)$

$$= \int_{\mathbb{R}^n} (\partial^\alpha f)(x)\, \underline{\underline{e}}^{-2\pi i\langle x,\xi\rangle}\, dx$$

$\int_{\mathbb{R}^n} f g'\, dx$

$= -\int_{\mathbb{R}^n} f' g\, dx$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) (-2\pi i \xi)^\alpha e^{-2\pi i\langle x,\xi\rangle} dx + (\text{boundary terms})$$

$$= (2\pi i \xi)^\alpha \hat{f}(\xi)$$

(2) Let $\alpha = e_j = (0, \ldots, 0, \underset{j^{th} \text{ coordinate}}{1}, 0, \ldots, 0)$

$$\frac{e^{-2\pi i\langle x, \xi + h e_j\rangle} - e^{-2\pi i\langle x, \xi\rangle}}{h} - (2\pi i x_j) e^{-2\pi i\langle x, \xi\rangle} \longrightarrow 0 \text{ as } h \to 0$$

bounded $\underline{C|x|}$ $\forall h$.

Use DCT with respect to the measure $\underline{f(x)\,dx}$.

$$(\partial_j \hat{f})(\xi) = \int_{\mathbb{R}^n} f(x) \left(\frac{e^{-2\pi i\langle x, \xi + h e_j\rangle} - e^{-2\pi i\langle x, \xi\rangle}}{h} \right) dx$$

fast

$\left(f \le \frac{1}{(1+|x|)^{n+1}} \right)$

$\to -(2\pi i x_j) e^{-2\pi i\langle x, \xi\rangle}$

$$= \int_{\mathbb{R}^n} \boxed{(-2\pi i x_j) f(x)} e^{-2\pi i \langle x, \xi \rangle} dx$$

$$= \left((-2\pi i x)^\alpha f(x) \right)^\wedge (\xi).$$

$\forall \alpha, \beta \in \mathbb{Z}_+^n.$

(3) $\| x^\alpha (\partial^\beta \hat{f}) \|_\infty$ $\rightsquigarrow$ $\left((-2\pi i x)^\beta f \right)^\wedge \rightarrow \partial$

$$= \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \| (\partial^\alpha (x^\beta f))^\wedge \|_\infty$$

$(\|\hat{f}\|_\infty \le \|f\|_1)$

$$\le (2\pi)^{|\beta|-|\alpha|} \| \partial^\alpha (x^\beta f) \|_{L^1} < \infty$$

(4) $\widehat{(f*g)}(\xi) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy\right) e^{-2\pi i\langle x,\xi\rangle}\, dx$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)\, g(y)\, e^{-2\pi i\langle x-y,\xi\rangle} \cdot e^{-2\pi i\langle y,\xi\rangle}\, dx\, dy$$

Fubini's theorem

$$= \left(\int_{\mathbb{R}^n} f(x\cancel{-y})\, e^{-2\pi i\langle x\cancel{-y},\xi\rangle}\, dx\right)\left(\int_{\mathbb{R}^n} g(y)\, e^{-2\pi i\langle y,\xi\rangle}\, dy\right)$$

$$= \hat{f}(\xi)\cdot\hat{g}(\xi).$$

(5) $O : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$

$$f(x) \mapsto f(Ox).$$

$$\boxed{\mathcal{F} \circ O = O \circ \mathcal{F}.}$$

$$(\mathcal{F} \circ O)(f)(\xi) = \widehat{f \circ O}(\xi) = \int_{\mathbb{R}^n} f(Ox)\, e^{-2\pi i \langle x, \xi \rangle}\, dx$$

$$(x \to O^T x)$$

$$= \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle O^T x, \xi \rangle}\, dx$$

$$= \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, O\xi \rangle}\, dx$$

$$= \hat{f}(O\xi)$$

$$= (O \circ \mathcal{F})(f)(\xi).$$

<u>Corollary</u>: The Fourier transform of a radially symmetric function is <u>radially symmetric</u>.

Pf. f is radially symmetric of a and only if

iff $\quad O(f) = f \quad \forall\, O \in$ orthogonal group (n).

$\mathcal{F}(f)$

$$O \circ (\mathcal{F}(f)) = \mathcal{F}(O(f)) = \mathcal{F}(f)$$

Remark. The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$!

The Inverse Fourier Transform.

Def^n. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We define

$$f^\vee(x) := \hat{f}(-x) \quad \forall\, x \in \mathbb{R}^n.$$

$$\left(f^\vee = f \right)$$

The operator $f \to f^\vee$ is called the <u>inverse Fourier transform</u>.

Theorem (Multiplication Formula) (very important)

For $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f(x)\, \hat{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(x)\, g(x)\, dx$$

$$\langle f, \overline{\hat{g}} \rangle = \langle \hat{f}, \overline{g} \rangle \qquad \widehat{\overline{g}} = \tilde{\hat{g}}$$

$\tilde{g} \to g$

Proof:

$$\int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} g(y)\, e^{-2\pi i \langle y, x \rangle}\, dy \right) dx$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, y \rangle}\, g(y)\, dx\, dy$$

$$\langle \hat{f}, g \rangle = \langle f, \widehat{\overline{g}} \rangle = \langle f, \tilde{\hat{g}} \rangle$$

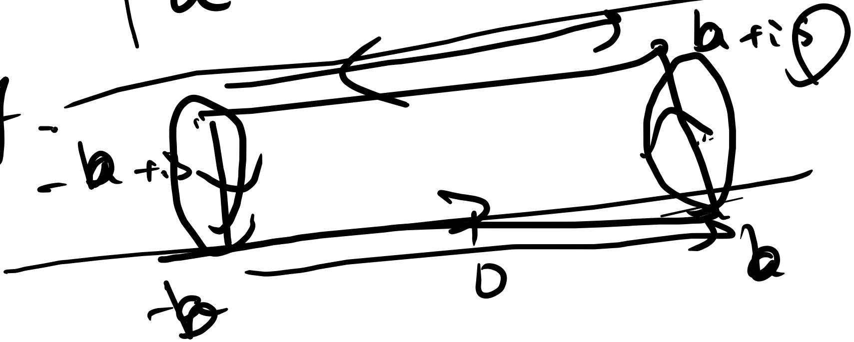
Lemma: For all $a > 0$, we have

$$\boxed{\int_{\mathbb{R}^n} e^{-\pi a |x|^2} \cdot e^{-2\pi i \langle x, \xi \rangle} \, dx} = \boxed{a^{-n/2} \, e^{-\pi |\xi|^2 / a}}.$$

$\left(e^{-|x|^2} = e^{-(x_1^2 + \cdots + x_n^2)} \in \mathcal{S}(\mathbb{R}^n) \right).$

- For $a = 1$: $\left(e^{-\pi |x|^2} \right)^\wedge (\xi) = e^{-\pi |\xi|^2}$.

Pf:



e^{-z^2} is an entire function.

$$\boxed{\mathbb{E}\left[e^{itX}\right]}$$

$(f^\vee \simeq \hat{f})$

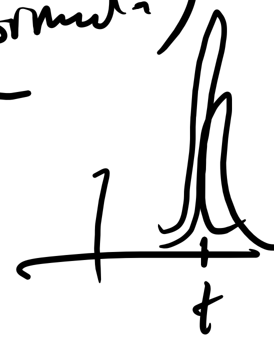
Theorem: $f \in S(\mathbb{R}^n)$

$$(\hat{f})^\vee = f = (f^\vee)^\wedge$$

(Fourier inversion formula)

Pf: Let $g_{t,\varepsilon}(\xi) = \boxed{e^{2\pi i\langle \xi, t\rangle}}\, e^{-\pi|\varepsilon \xi|^2}$

$$\hat{g}_{t,\varepsilon}(x) = \frac{1}{\varepsilon^n}\, e^{-\pi\left|\frac{x-t}{\varepsilon}\right|^2}$$



$$\int_{\mathbb{R}^n} f(x) \left[\frac{1}{\varepsilon^n} e^{-\pi \left|\frac{x-t}{\varepsilon}\right|^2}\right] dx$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi)\, g_{t,\varepsilon}(\xi)\, d\xi.$$

$\| f * e_{\varepsilon} - f \|_\infty \to 0$

When $\varepsilon \to 0$, LHS $\to f$ uniformly.

RHS $\longrightarrow (\hat{f})^{\vee}(t)$ as $\varepsilon \to 0$

(by DCT)

$\int_{\mathbb{R}^n} \hat{f}(\xi) \cdot e^{2\pi i \langle \xi, t \rangle} d\xi$

Thus, $(\hat{f})^{\vee} = f$ on $\mathbb{R}^n$.

Replacing f by $\tilde{f}$, we have

$$(f^{\vee})^{\wedge} = f.$$

$$f^{\vee}(\xi) = \hat{\tilde{f}}(\xi) = \int_{\mathbb{R}^n} \tilde{f}(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

$$f^{\vee}(\xi) = \hat{\tilde{f}}(\xi) = \hat{f}(-\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x, \xi \rangle} dx$$

$$f^\vee(x) = \int_{\mathbb{R}^n} f(y)\, e^{2\pi i \langle x, y\rangle}\, dy$$

(Fourier inversion formula.)

$$(f^\vee)^\wedge = f, \quad (f^\wedge)^\vee = f. \qquad \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

$$\longleftarrow (\cdot)^\vee$$

Theorem (1) (Parseval's formula)

$$\int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,dx$$

Pf: $\left(\overline{\hat{g}}\right)^{\wedge} = \overline{g}$.

(2) (Plancherel's identity)

$$\int_{\mathbb{R}^n} |f(x)|^2\,dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi.$$

(3) $$\int_{\mathbb{R}^n} f(x)\cdot \hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x)\cdot (g(x))\,dx$$

Corollary: The Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself.

Pf. $f_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ $\qquad \|\hat{f}_n\| \to 0$

$\|x^\alpha \partial^\beta f_n\|_\infty \to 0$

$\left[f_n \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n) \right.$

$\Rightarrow f_n \to 0 \text{ in } L^1(\mathbb{R}^n)$, $\quad x^\alpha \partial^\beta f \to 0 \text{ in } L^1(\mathbb{R}^n)$

$\Rightarrow \hat{f}_n \to 0 \text{ in } L^\infty(\mathbb{R}^n)$, $\quad x^\alpha \partial^\beta \hat{f} \to 0 \text{ in } L^\infty(\mathbb{R}^n)$

$\partial^\alpha x^\beta \hat{f} \to 0 \text{ in } L^\infty(\mathbb{R}^n)$ $\quad \|\hat{f}\|_\infty \le \|f\|_1 \to 0$