31/07/2021 Lecture - 4

$f_k \to f$ in $L^2(S^1)$

$\hat{f}_k \in \ell^2(\mathbb{Z})$ and is convergent.

$\hat{f}_k \longrightarrow g \in \ell^2(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})$

$L^2(S^1) \subseteq L^1(S^1)$.

$f \in L^1(S^1)$. To show: $g = \hat{f} \in \ell^\infty(\mathbb{Z})$.

$\ell^2(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})$

$\mathcal{F} : C^\infty(S^1) \longrightarrow \mathcal{S}(\mathbb{Z})$

smooth function on S^1

rapidly decreasing function on $\mathbb{Z}$.

$$f = \sum_{k \in \mathbb{Z}} c_k(f)\, e^{2\pi i k x}$$

$$f' = (2\pi i) \sum_{k \in \mathbb{Z}} k\, c_k(f)\, e^{2\pi i k x}$$

The Class of Schwartz functions on $\mathbb{R}^n$

A function is Schwartz if it is smooth and uniformly bounded and so are its derivatives, ~~and~~ the function and its derivatives decay faster than the reciprocal of any polynomial.

Notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_+)^n$ multi-index

$$\partial^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

Defn: A C^∞ complex-valued function f on $\mathbb{R}^n$ is called a *Schwartz function* if for every pair of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\exists$ a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty$$

$x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Schwartz seminorm.

Set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $S(\mathbb{R}^n)$.

Example: (1) $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$

$$\hookrightarrow e^{-(x_1^2 + \cdots + x_n^2)}$$

(2) The set of all smooth function with compact support is contained in $\mathcal{S}(\mathbb{R}^n)$.

$$C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n).$$

Remarks:

(1) $f(x_1, \ldots, x_{n+m}) = f_1(x_1, \ldots, x_n) \cdot f_2(x_{n+1}, \ldots, x_{m+n})$

If $f_1 \in \mathcal{S}(\mathbb{R}^n)$, $f_2 \in \mathcal{S}(\mathbb{R}^m)$,

then f is a Schwartz function on $\mathbb{R}^{n+m}$.

(2) If $f \in \mathcal{S}(\mathbb{R}^n)$, and $P(x)$ is a polynomial in n variables, then $Pf \in \mathcal{S}(\mathbb{R}^n)$.

(3) If α is a multiindex, $f \in \mathcal{S}(\mathbb{R}^n)$, then

$\underline{\underline{\partial^\alpha f}} \in \mathcal{S}(\mathbb{R}^n)$.

$f \in \mathcal{S}(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n} |\partial^{\alpha}(x^{\beta} f)| < \infty$

$\forall$ multiindices α, β.

Remark: A C^∞-function f is in $\mathcal{S}(\mathbb{R}^n)$ if and only if $\forall N \geq 0$, and all $\alpha \in \mathbb{Z}_+^n$, $\exists$ $C_{\boxed{\alpha, N}}$, such that

$$|\partial^{\alpha} f(x)| \leq C_{\alpha, N} \frac{1}{(1+|x|)^N}$$

$\mathcal{S}(\mathbb{R}^n) \longrightarrow$ vector space over $\mathbb{C}$.

Defn: Let (f_k) be a sequence in $\mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We say that $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$ if $\forall\ \alpha, \beta \in \mathbb{Z}_+^n$ we have

$$\underline{\underline{\rho_{\alpha,\beta}}}(f_k - f) \longrightarrow 0 \text{ as } k \to \infty.$$

$$\sup_{x \in \mathbb{R}^n} |x^\alpha (\partial^\beta (f_k - f))(x)| \to 0.$$

For instance, $f(x + \frac{1}{k}) \longrightarrow f(x)$ in $\mathcal{S}(\mathbb{R}^n)$ for all f in $\mathcal{S}(\mathbb{R}^n)$ as $k \to \infty$.

The Schwartz topology on $\boxed{\mathcal{S}(\mathbb{R}^n)}$ is the translation-invariant topology defined by the following sub-basis of open neighborhoods of 0.

$$N_{\alpha,\beta,\varepsilon}(0) = \{ f \in \mathcal{S}(\mathbb{R}^n) : \rho_{\alpha,\beta}(f) < \varepsilon \}.$$

$\forall\ \alpha, \beta \in \mathbb{Z}_+^n$, and $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$

Remarks on the Schwartz topology:

(1) The mappings, $+ : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

$(f, g) \mapsto f + g$

is $\cdot : \mathbb{C} \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$; $(a, f) \mapsto af$

$$\partial^\alpha : \boxed{S(\mathbb{R}^n)} \to S(\mathbb{R}^n)$$
$$f \to \boxed{\partial^\alpha f}$$

are continuous in the Schwartz topology..

(II) If $\rho_{\alpha,\beta}(f) = 0 \quad \forall\, \alpha, \beta \in \mathbb{Z}_+^n$, then $f = 0$.. ($\frac{df}{dx} = 0 \Rightarrow f = \text{constant}$)

$\Rightarrow f = 0$ because of the decay property

(III) $d(f,g) = \displaystyle\sum_{j=1}^{\infty} 2^{-j} \boxed{\frac{\rho_j(f-g)}{1 + \rho_j(f-g)}}$ defn..

a metric on $\mathcal{S}(\mathbb{R}^n)$, and the corresponding metric topology is the Schwartz topology. j — reindexing of the countable set $\mathbb{Z}_+^n \times \mathbb{Z}_+^n$.

(iv) $\mathcal{S}(\mathbb{R}^n)$ is <u>complete</u>. If (h_k) is Cauchy in $\mathcal{S}(\mathbb{R}^n)$, then it is Cauchy in ~~L^∞~~ sup norm, and hence converges uniformly to some continuous function h. $\Rightarrow$ Same holds for $(\partial^\beta h_k)$ and $(\underline{x^\alpha} h_k)$. $\boxed{\partial^\beta h_k \to \partial^\beta \underline{h}}$.

$\therefore$ $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space.

(complete metrizable locally convex space)

Proposition: Let (f_k) be a sequence in $\mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. If $f_k \to f$ (as $k \to \infty$) in the Schwartz topology, then $\partial^\beta f_k \to \partial^\beta f$ in L^p, $\forall\, 0 < p \le \infty$, $\forall\, \beta \in \mathbb{Z}_+^n$.

$\|a+b\| \le C(\|a\| + \|b\|)$

Pf: $\|\partial^\beta f\|_p = \left(\int_{|x| \le 1} |\partial^\beta f(x)|^p \, dx + \int_{|x| > 1} \boxed{|x|^{n+1} |\partial^\beta f(x)|}^p |x|^{-(n+1)} \, dx \right)^{1/p}$

$$\leq \Big(\mathrm{vol}(B_1) \|\partial^\beta f\|_\infty^p + \big(\sup_{|x|\geq 1} |x|^{n+1} |\partial^\beta f(x)|\big)^p \int_{|x|\geq 1} |x|^{-p(n+1)}\, dx \Big)^{1/p}$$

$$\int_1^\infty \frac{1}{x^2}\, dx$$

(Minkowski's inequality.)

$$\leq \boxed{C_{p,n}} \Big(\|\partial^\beta f\|_\infty + \sup_{|x|\geq 1} |x|^{\left[\frac{n+1}{p}\right]+1} \cdot |\partial^\beta f(x)| \Big)$$

$$m := \left[\frac{n+1}{p}\right]+1$$

$$\Big[\, |x|^m \boxed{|\partial^\beta f(x)|} \leq C_{n,m} \Big(\sum_{|\alpha|=m} |x^\alpha \boxed{\partial^\beta f(x)}| \Big) \Big]$$

$(x_1, \ldots, x_n)$

$$\left(|x|^k \le C_{n,k} \left(\sum_{|\beta|=k} |x^\beta| \right) \right),$$

$$x \mapsto \boxed{\sum_{|\beta|=k} \frac{|x^\beta|}{|x|^k}}$$ has a strictly positive minimum on $\boxed{S^{n-1}}$.

$x_1^{\beta_1}=0 \Rightarrow x_1=0$

$x_2^{\beta_2}=0 \Rightarrow x_2=0$

$$\frac{1}{C_{n,k}} \le \sum_{|\beta|=k} |x^\beta| \quad \text{on } S^{n-1}.$$

Thus the L^p-norm of $\partial^\beta f$ is controlled by a constant multiple of some $\rho_{\alpha,0}$ seminorms of $\partial^\beta f$.

Proposition. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then fg, $f * g \in \mathcal{S}(\mathbb{R}^n)$. Moreover,

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$$

Pf. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$.

j^{th} position

$$\frac{f(y + h e_j) - f(y)}{h} - \partial_j f(y) \to 0, \text{ as } h \to 0$$

uniformly with respect to y.

$(a^p + b^p)^{1/p} \leq a + b$, $1 \leq p$.

$(a^p + b^p)^{1/p} \leq \boxed{C_p} (a + b)$, $0 < p \leq 1$

$$\left| \int_0^h (h-t) \, \boxed{\partial_j^2 f(y+te_j)} \, dt \right|$$

$$\le \frac{M h^2}{2} \qquad \boxed{M\max}$$

By DCT with respect to the measure $\boxed{g(x-y)\,dy}$.

$$\int_{\mathbb{R}} \left(\frac{f(y+he_j)-f(y)}{h} - \partial_j f(y) \right) g(x-y)\,dy \longrightarrow 0$$

$$= \int_{\mathbb{R}} \frac{f(y+he_j) - f(y)}{h}\, g(x-y)\, dy$$

$$\longrightarrow \int_{\mathbb{R}} (\partial_j f)(y)\, g(x-y)\, dy$$

$$= \boxed{((\partial_j f) * g)(x)}$$

$$\lim_{h\to 0} \frac{(f*g)(x+he_j) - (f*g)(x)}{h}$$

$$= \frac{\partial}{\partial x_j}(f*g)\Big|(x) \qquad \partial_j(f*g) = (\partial_j f) * g$$

Use induction to prove the full formula.

$f * g \in C^\infty(\mathbb{R}^n)$, $N \geq n+1$. $\boxed{\int \frac{1}{(1+|x|)^{n+1}} dx < \infty}$

$|(f*g)(x)|$

$\leq \boxed{C_N} \int_{\mathbb{R}^n} \frac{1}{(1+|x-y|)^N} \frac{1}{(1+|y|)^N} dy$

$= I_1 + I_2$

$I_2 \to |y-x| \leq \frac{1}{2}|x| \Rightarrow |y| \leq \frac{3}{2}|x|$.

$I_1 \to |y-x| \geq \frac{1}{2}|x|$ ~~$\Rightarrow |y| + |x| \geq \frac{1}{2}|x|$~~

$$I_1 \le \underline{\int_{|y-x| \geq \frac{1}{2}|x|}} \boxed{\frac{1}{(1+\frac{1}{2}|x|)^N}} \frac{1}{(1+|y|)^N} dy$$

$$\le B_N (1+|x|)^{-N}$$

$$I_2 \le \int_{|y-x| \le \frac{1}{2}|x|} \frac{1}{(1+|x-y|)^N} \frac{1}{(1+\frac{1}{2}|x|)^N} dx$$

$|x|-|y| \le \frac{1}{2}|x|$

$|y| \geq \frac{1}{2}|x|$

$$\le \boxed{B'_N} (1+|x|)^{-N}$$

$f * g$ decays like $(1+|x|)^{-N}$ at infinity, for arbitrary $N > n$. ($N \geq n+1$).

Thus, $f * g$ decays faster than the reciprocal of any polynomial.

$f * g \in \mathcal{S}(\mathbb{R}^n)$ (Use Leibnitz rule repeatedly)

$$\boxed{f = \sum_{k \in \mathbb{Z}} c_k(f)\, e^{2\pi i k x}}.$$

$f_k \to f$ in $\boxed{\mathcal{S}(\mathbb{R}^n)}$

$\Rightarrow \partial^\beta f_k \to \partial^\beta f$ in $L^p(\mathbb{R}^n)$, $\forall p \in (1, \infty]$

[Classical Fourier analysis → Loukas Grafakos]

$$\sum |x_i - x_{i-1}| < \delta$$

$$\sum |f(x_i) - f(x_{i-1})| < \varepsilon$$