24/07/2021 (Analysis - 2)

$\hat{\cdot} \,;\, L^1(S^1, m) \longrightarrow c_0(\mathbb{Z})$

$\left[\ \hat{\cdot} \,;\, C^\infty(S^1) \longrightarrow \mathcal{S}(\mathbb{Z})\ \right]$

$\mathcal{S}(\mathbb{Z}) \to$ rapidly decreasing functions on $\mathbb{Z}$

$f \stackrel{?}{=} \sum_{k \in \mathbb{Z}} c_k(f) e_k$

$f \in L^2(S^1, m)$, $\ell^2(\mathbb{Z})$.

$L^1(S^1)$

$\supseteq L^p(S^1)$, $\forall_k p \leq \infty$

$L^r(S^1) \supseteq L^s(S^1)$

$1 \leq r \leq s \leq \infty$.

Pre-Hilbert and Hilbert spaces

Defn: A complex vector space V together with an inner product $\langle \cdot, \cdot \rangle$ is called a <u>pre-Hilbert space</u> or <u>inner product space</u> -

$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is an inner product:

(i) $\langle \cdot, \cdot \rangle$ is linear in first entry and conjugate-linear in the <u>2nd</u> entry.

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

(ii) $\langle \cdot, \cdot \rangle$ is positive-definite; $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0 \iff v = 0$.

Examples: (I) $V = \mathbb{C}$, $\langle \alpha, \beta \rangle = \alpha \bar{\beta}$

(II) $V = \mathbb{C}^k$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$

$$\langle v, w \rangle = v_1 \overline{w_1} + \cdots + v_k \overline{w_k}.$$

$$\|v\| := \sqrt{\langle v, v \rangle} \quad \text{for } v \in V.$$

Lemma (Cauchy Schwarz inequality)

$V \to$ pre-Hilbert space. Then for $v, w \in V$

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

This implies that v is a norm, that is,

(1) $\|\lambda v\| = |\lambda| \cdot \|v\| \quad \forall \lambda \in \mathbb{C}$, (positive homogeneity)

(2) $\|v\| \geq 0$; $\|v\| = 0 \iff v = 0$ (positive-definiteness)

(3) $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

Pf: $a(t) := \langle v + tw, v + tw \rangle$, $a : \mathbb{R} \to \mathbb{R}_+$

$$a(t) = \|v\|^2 + t^2 \|w\|^2 + t \underbrace{(\langle v, w\rangle + \langle w, v\rangle)}_{2\mathrm{Re}\langle v, w\rangle}$$

a attains minimum at $t_0 = \boxed{-\dfrac{\mathrm{Re}\langle v, w\rangle}{\|w\|^2}}$

$$0 \leq a(t_0) = \|v\|^2 + \frac{(Re\langle v, w\rangle)^2}{\|w\|^4} \cdot \|w\|^2$$

$$-2\frac{Re\langle v, w\rangle}{\|w\|^2}$$

$$\Rightarrow (Re\langle v, w\rangle)^2 \leq \|v\|^2 \cdot \|w\|^2$$

Replace v with $\frac{|\langle v, w\rangle|}{\langle v, w\rangle} v$ (if $\langle v, w\rangle \neq 0$)

$i\,v$ (if $\langle v, w\rangle = 0$)

$$\Rightarrow |\langle v, w\rangle|^2 \leq \|v\|^2 \cdot \|w\|^2$$

$$\|v+w\|^2 = \langle v+w, v+w\rangle$$

$$= \|v\|^2 + \|w\|^2 + 2\,\mathrm{Re}\,\langle v, w\rangle$$

$$\leq \|v\|^2 + \|w\|^2 + 2\,|\langle v, w\rangle|$$

$$\overset{(C\text{-}S)}{\leq} \|v\|^2 + \|w\|^2 + 2\,\|v\|\,\|w\|$$

$$= \left(\|v\| + \|w\|\right)^2$$

$$\Rightarrow \boxed{\|v+w\| \leq \|v\| + \|w\|}$$

$$\|v\| := \sqrt{\langle v, v\rangle}$$

Corollary: $\big|\,\|v\| - \|w\|\,\big| \leq \|v-w\|$

Pf: $\|v\| \leq \|w\| + \|v-w\|$

$\|w\| \leq \|v\| + \|w-v\|$

$\blacksquare$

Def^n: A linear map $T: V \to W$ between pre-Hilbert spaces V, W is said to be an <u>isometry</u> if T preserves the norm,

$\|Tv\| = \|v\|$, for any $v \in V$.

(equivalently, $\langle Tv, Tv' \rangle = \langle v, v' \rangle \ \forall v, v' \in V$)

Clearly, T is injective. (If $Tv = 0 \Rightarrow \|Tv\| = 0$
$\Rightarrow \|v\| = 0 \Rightarrow v = 0$)

But ~~T~~ an isometry
T need not be surjective.

If T is surjective, then T is said to be a unitary (unitary = surjective isometries).

Def^n: A pre-Hilbert space that is complete under the norm $\|\cdot\|$, is said to be a Hilbert space.

l^2 space:

Let S be a set. $l^2(S)$ → set of functions on S, $f: S \to \mathbb{C}$, such that

$$\sum_{s \in S} |f(s)|^2 < \infty$$

(All but countably many ~~of the~~ points in S must map to 0.). $\left[\left\{a_k : \frac{1}{n} \le a_k\right\} = A_n\right\} \subseteq \{a_k\}$.

$$\|f\|_2^2 = \sum_{s \in S} |f(s)|^2 = \sup_{\substack{F \subset S \\ F \text{ finite}}} \left(\sum_{s \in F} |f(s)|^2 \right)$$

Theorem: Let S be any set. Then $\ell^2(S)$ forms a Hilbert space with inner product,

$$\langle f, g\rangle = \left(\sum_{s\in S} f(s)\cdot\overline{g(s)}\right), \quad f, g \in \ell^2(S).$$

$F_1,$ $\boxed{\sum_{s\in F_1} h(s)}$

$F_1 \subseteq F_2$

Pf: Assume f, g take values in $\mathbb{R}_+$.

$$\langle f, g\rangle = \sup_{\substack{F\subset S\\ F \text{ finite}}}\left(\sum_{s\in F} f(s)\cdot g(s)\right)$$

$$\leq \sup_{\substack{F\subset S\\ F \text{ finite}}}\left(\left(\sum_{s\in F} f(s)^2\right)^{1/2}\left(\sum_{s\in F} g(s)^2\right)^{1/2}\right)$$

$$\leq \sup_{\substack{F\subset S\\ F \text{ fin...}}} \|f\|_2^{\,2}\, \|g\|_2^{\,2}$$

$\|f\|_2 \cdot \|g\|_2$

$f = u + iv$

$u = u_+ - u_-$

positive real-valued.

$u_+ = \max\{u, 0\}$

$u_- = -\min\{u, 0\}$.

$$\langle f, g\rangle = \sum_{s\in S} f(s)\overline{g(s)}.$$

Completeness: Let $s_0 \in S$. If f_n is Cauchy in ℓ^2-norm, then $(f_n(s_0))$ is Cauchy in $\mathbb{C}$. Hence, $f_n(s_0) \to f(s_0)$ for some $f(s_0) \in \mathbb{C}$.

$$\sup_{\substack{F \subset S \\ F \text{ finite}}} \sum_{s \in F} |f_n(s) - f(s)|^2 = \lim_{j \to \infty} \left(\sum_{s \in F} |f_n(s) - f_j(s)|^2 \right)$$

$$\leq \sup_{j \geq N} \|f_n - f_j\|_2^2 < \varepsilon.$$

Let $\varepsilon > 0$, $N \in \mathbb{N}$, s.t. $\forall\, m, n \geq N$

$\|f_n - f_m\|_2^2 < \varepsilon$. For finite F.

$\Rightarrow$ $f_n \to f$ in the ℓ^2 norm.

Orthonormal basis and completeness

Defn: A <u>complete system</u> in a pre-Hilbert space H is a family $(v_j)_{j \in J}$ of vectors in H such that $\mathrm{Span}\,(v_j)_{j \in J}$ is dense in H.

H is said to be <u>separable</u> if there is a <u>countable</u> complete system of vectors.

Example:- (1) If H is finite-dimensional, any set that contains a basis is a **complete system**.

(2) $\ell^2(\mathbb{N})$ → square summable sequences.

$$\psi_j(k) = \delta_{kj} = \begin{cases} 0, & \text{if } k \neq j \\ 1, & \text{if } k = j \end{cases}$$

$$\langle f, \psi_j \rangle = \underline{f(j)}$$

$$\boxed{\sum_{k=1}^{n} f(j)\,\psi_j(\cdot)} \to f$$

in the ℓ^2.

$\{\psi_j\}$ is a complete system.

0, 0, 0, ...

sum of the truncation of f tends to 0 in ℓ^2.

Defn. An <u>orthonormal system</u> in a pre-Hilbert space H is a family $(h_j)_{j \in J}$ of vectors in H such that $\langle h_j, h_{j'} \rangle = \delta_{j,j'}$ for $j, j' \in H$.

$$\left(\|h_j\| = 1 \quad \forall j \in H \right).$$

An orthonormal system which is also a complete system is called an orthonormal basis

(Frames).

$$\langle \psi_j, \psi_{j'} \rangle = \sum_{s \in \mathbb{N}} \underline{\underline{\psi_j(s)}} \cdot \underline{\underline{\overline{\psi_{j'}(s)}}} \quad \mathbb{N} \setminus \{j, j'\}$$

$$= \delta_{j,j'}$$

$\{\psi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $\ell^2(\mathbb{N})$.

<u>Proposition</u>: Every separable pre-Hilbert space admits an orthonormal basis.

Pf: Use induction $\boxed{\{u_1, \dots, u_n\}} \cup \{u_{n+1}\}$

Theorem. ~~If~~ Let H be an infinite-dimensional separable pre-Hilbert space and let $(e_j)_{j \in J}$ be an orthonormal basis of H. Then every element $h \in H$ can be represented in the form

$$h \overset{?}{=} \sum_{j=1}^{\infty} c_j e_j \quad \text{, where the convergence}$$

is in H, and the ~~coeff~~ coefficients c_j satisfy

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty \qquad (c_j = c_j(h) = \langle h, e_j \rangle).$$

$h \longrightarrow (c_j)_{j \in \mathbb{N}}$ gives an isometry from H to $\ell^2(\mathbb{N})$.

In particular, $\|h\|^2 \geq \sum_{j=1}^{\infty} |c_j|^2$.

Pf: $S_n(h) = \sum_{j=1}^{n} c_j e_j \in H$.

$$0 \leq \|h - S_n(h)\|^2 = \langle h, h\rangle - \langle h, S_n(h)\rangle - \langle S_n(h), h\rangle + \|S_n(h)\|^2$$

$$= \|h\|^2 - 2\sum_{j=1}^{n} |c_j|^2 + \sum_{j=1}^{n} |c_j|^2 \qquad \langle h, S_n(h)\rangle = \sum_{j=1}^{n} \overline{c_j}\langle h, e_j\rangle = \sum_{j=1}^{n} |c_j|^2$$

$$= \|h\|^2 - \sum_{j=1}^{n} |c_j|^2$$

$$\sum_{j=1}^{n} |c_j|^2 \leq \|h\|^2$$

$$\Rightarrow \boxed{\lim_{n\to\infty} |c_j|^2 \leq \|h\|^2}$$

$$\boxed{(c_j)_{j\in J} \in l^2(J).}$$

$$\boxed{h = \sum_{j=1}^{\infty} c_j e_j} \text{ . (completeness).}$$

Theorem. The exponentials $e_k(x) = e^{2\pi i k x}$, $k \in \mathbb{Z}$ form an orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$ $(L^2(S^1, m))$.

Pf: (e_k) is an orthonormal system.

$$\langle e_k, e_l \rangle = \int_0^1 e_k \overline{e_l} \, dm(x) = \int_0^1 e^{2\pi i (k-l) x} \, dx = \delta_{k,l}$$

Completeness: [$\langle f, e_k \rangle = 0$ for all $k \in \mathbb{Z}$]

Since $f \in L^2(S^1, m) \Rightarrow f \in L^1(S^1, m)$

$\hat{f} = 0 \Rightarrow f = 0$

Theorem: Let $f \in L^2(S^1)$. Then

$$\|f\|_{L^2}^2 := \int_0^1 |f(x)|^2 \, dx = \sum_{k\in\mathbb{Z}} |c_k|^2 =: \|(c_k)_{k\in\mathbb{Z}}\|_{\ell^2}^2$$

$\in \ell^2(\mathbb{Z})$.

Pf: Corollary of previous theorem.

$$\mathcal{F} := \hat{\cdot} : L^2(S^1) \longrightarrow \ell_0(\mathbb{Z}).$$

$$\boxed{\mathcal{F}\left(L^2(S^1, m)\right) = \ell^2(\mathbb{Z})} \subsetneq \ell_0(\mathbb{Z}).$$

unitary operator.

This is also known as [Plancherel's theorem]

Poisson summation formula:

discretization of a function
(uniform sampling)

$\Longleftrightarrow$ periodization of a function

$\boxed{F_n \to \delta}$

$f * F_n \to f * \delta$

$\boxed{u^{\mathcal{F}} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}}$

$\delta_0 : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$

$f \to f(0)$

$\boxed{\mathcal{F} : \boxed{C^\infty(\mathbb{T})} \to \mathcal{S}(\mathbb{Z})}$

$L^2 \to l^2$

$L^2 \to l^2$

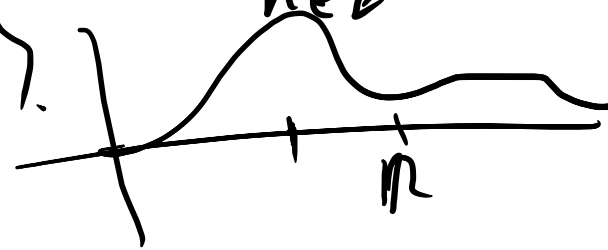
f : → continuous function on $\mathbb{R}$

$g(x) = \sum_{n \in \mathbb{Z}} f(x+n)$

periodization...

discrete subgroup of S^1

$= \{ e^{2\pi i m / N} ; 0 \le m < N \}$



$$g(x) := \frac{1}{N} \sum_{k=0}^{N-1} f\left(x + \frac{k}{N}\right)$$

$$g\left(x + \frac{1}{N}\right) = \frac{1}{N} \sum_{k=0}^{N-1} f\left(x + \frac{(k+1)}{N}\right)$$

$= g(x).$

periodization on S^1

uniform sampling on $\mathbb{Z}$

$$g(x) = \frac{1}{N} \sum_{k=0}^{N-1} f\left(x + \frac{k}{N}\right),$$

$\omega = N^{th}$ root of unity

$\frac{1}{N}(1 + \omega + \omega^2 + \dots + \omega^{N-1})$

$= 1$ if $\omega = 1$

$= 0$ otherwise

$$c_k(\boxed{T_a} \circ f) = \int_0^1 f(x+a) \cdot e^{-2\pi i k x}\, dx \int f(x)$$

$$= \boxed{e^{2\pi i k a}} \cdot c_k(f).$$

$$\boxed{c_k(g)} = c_k(f)\left(\frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi i \frac{km}{N}}\right)$$

$= 1$ if $N \mid k$

$= 0$ if $N \nmid k$.

$$1 + x + \cdots + x^{N-1} = \frac{1 - x^N}{1 - x} = 0 \quad \text{if } x \neq 1 \text{ and } x^N = 1$$

$C_n(f) = C_k(f)$ if $N \mid k$

0 if $N \nmid k$

$\odot$: N

0

2π

$$\mathcal{F}\left(\frac{1}{N}\sum_{k=0}^{N-1} f\left(x+\frac{k}{N}\right)\right)$$

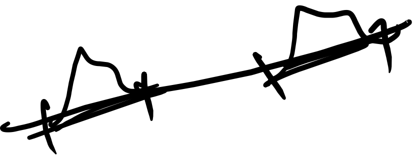
$=$ sampling of $\mathcal{F}(f)$

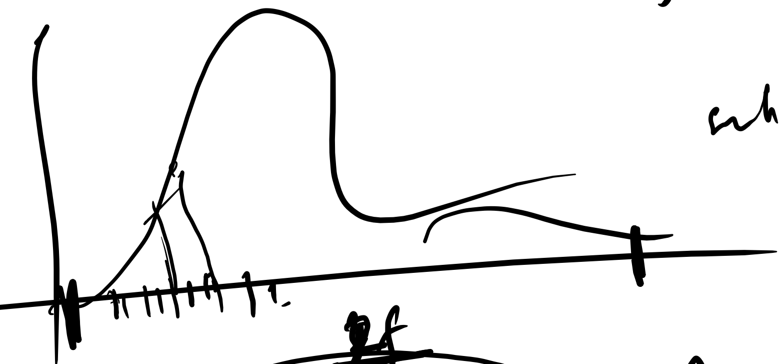
$\cdots \quad 1 \quad 0 \quad \cdot \quad \cdot \quad N-2$

~~Sham~~

Shannon - Nyquist theorem

$f \in L^2$ is a function on $\mathbb{R}$ such that $\hat{f}$ is compactly supported.





$2f$

band - limited $\equiv$ $\hat{f}$ is compactly supported.

$\boxed{2f}$

$2N$

N