20/07/2021:

$$\hat{\cdot} : L^1(S^1, m) \longrightarrow l_0(\mathbb{Z})$$

$$f \longrightarrow \hat{f}$$

$l_0(\mathbb{Z}) \hookrightarrow (f : \mathbb{Z} \to \mathbb{C})$ vanishing at infinity.

1) Is the Fourier transform injective?

2) What is the range of Fourier transform?

$C^\infty(S^1, m) \longleftrightarrow$ rapidly decaying sequence

3) How can we recover f from knowledge of $\hat{f}$:

$$f \overset{??}{=} \sum_{i \in S} a_i f_i$$

$\hookrightarrow \#(S) = \infty$.

__Lemma 1:__ If $k, l \in \mathbb{Z}$, $\left(e_k(x) := e^{2\pi i k x}\right)$

$$\int_0^1 e_k(x)\overline{e_l(x)}\,dx = \delta_{kl} \begin{cases} 0, & \text{if } k \neq l \\ 1, & \text{if } k = l \end{cases}$$

Pf: $\int_0^1 e_{k-l}(x)\,dx = \int_0^1 1\,dx$ (if $k = l$)

$= \int_0^1 e^{2\pi i s x}\,dx$ for some $s \in \mathbb{Z} - \{0\}$

$= 0$

$\hat{\cdot} : L^1(S^1, m) \to l_0(\mathbb{Z})$.

$$\boxed{f(x) = \sum_{k \in \{1,\dots,m\}} c_{n_k} e_{n_k}}$$

, $c_{n_k} \in \mathbb{C}$

$n_k \in \mathbb{Z}$

$1 \leq k \leq m$.

$$c_k(f) = \int_0^1 f(x) \cdot e^{-2\pi i k x}\, dx$$

$= 0$ if $k \notin \{n_1, \dots, n_m\}$

c_{n_j} if $k = n_j$.

Any cond of inverse of Fourier transform should look something like

$$f \sim \boxed{\sum_{n \in \mathbb{Z}} \underline{\underline{c_n(f)}}\, e_n}$$

$1 \le p \le \infty$

If $f \in L^1(S^1, m)$, $g \in L^p(S^1, m)$, then

$f * g \in L^p(S^1, m)$ and $\boxed{\|f * g\|_p \le \|f\|_1 \cdot \|g\|_p}$ $\longrightarrow$ $\|L_f(g)\|_p \le \|g\|_p$

$$L^1(S^1, m) \longrightarrow \mathcal{B}(L^p(S^1, m))$$

$$f \longrightarrow L_f \qquad (L_f(g) = f * g)$$

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \quad \left(c_k(f * g) = c_k(f) \cdot c_k(g) \right)$$

Assume $p = 1$. Is there an identity for the convolution algebra $L^1(S^1, m)$?

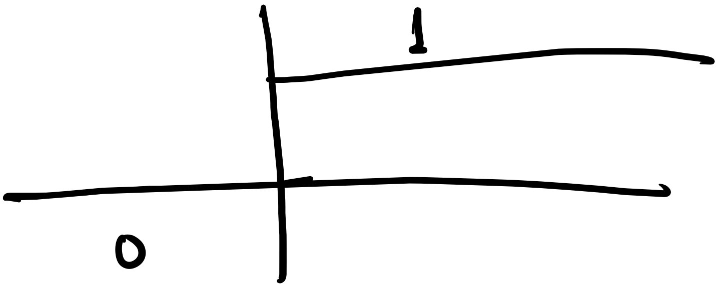
Let δ be that identity (if possible)

$$f * \delta = f \quad \forall f \in L^1(S^1, m)$$

$$c_k(f) \cdot c_k(\delta) = c_k(f) \quad \forall f \in L^1(S^1, \ldots), \forall k \in \mathbb{Z}.$$

$$\left[\boxed{c_k(\delta) = 1 \ \forall k \in \mathbb{Z}} \ \forall f \in L^1(S^1, m) \right].$$

$\delta_0 \to$ Dirac delta function.



Heaviside step function.

derivative of Heaviside = Dirac delta

δ_0 is such a function so that it is supported at one point '0', but it is so spiky that

$$\int_{-\infty}^{\infty} \delta_0(x)\, dx = 1$$

$$(f * \delta_0)(x) = \int_0^1 f(x-y)\,\underline{\underline{\delta_0}}(y)\,dy$$

$$= \cancel{\int_0} \int_0^1 \delta_0(x-y)\,f(y)\,dy$$

$$= f(x)$$

"Fourier transform" of the Delta function at 0 is the constant function 1.

Lemma: e_k is an eigenvector for L_f ($f \in L^1(S^1, m)$), with corresponding eigenvalue $c_k(f)$.

Pf: $L_f(e_k)(x) = f * e_k(x)$

$$= \int_0^1 f(x-y)\, e_k(y)\, dy$$

$$= \int_0^1 e_k(x-y) \cdot f(y)\, dy$$

$$= e_k(x) \underline{\underline{\int_0^1 f(y)\, e_k(-y)\, dy}}$$

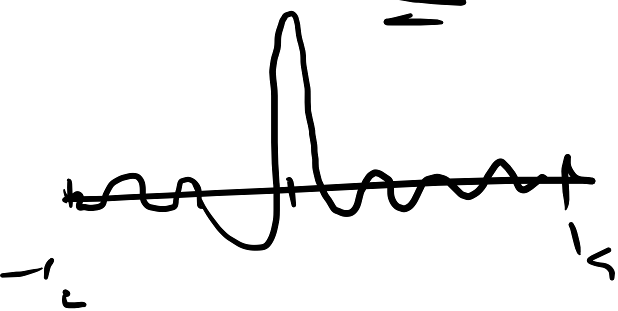
$$= e_k(x) \cdot c_k(f). \quad \forall x \in S^1.$$

Corollary: Let $D_n := \sum_{k=-n}^{n} e_k$. Then

$$f * \boxed{D_n} = \boxed{S_n(f)} \quad \left(:= \sum_{k=-n}^{n} c_k(f)\, e_k \right)$$

(D_n – Dirichlet kernel)

Pf: $\underline{\underline{L_f(D_n)}} = \sum_{k=-n}^{n} L_f(e_k) = \sum_{k=-n}^{n} c_k(f)\, e_k$



$$= \underline{\underline{S_n(f)}}.$$

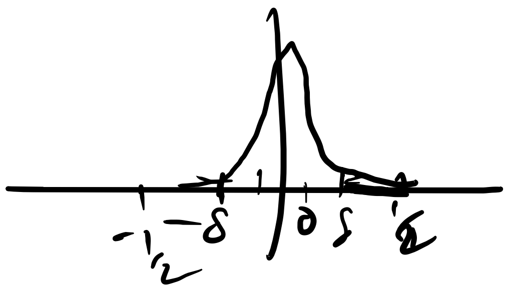
<u>Note</u>: $\int_0^1 D_n(t)\,dt = 1$, $\left(\int_0^1 |D_n(t)|\,dt \geq \Theta(\ln n)\right)$!

Let, $F_n := \frac{1}{n}\left(D_0 + D_1 + \dots + D_{n-1}\right)$

(Fejér kernel)

$$F_n(x) = \frac{1}{n}\left(\frac{\sin(n\pi x)}{\sin(\pi x)}\right)^2, \quad \forall x \in [0,1)$$

$$\int_0^1 F_n(t)\,dt = 1 \qquad \forall \; n \in \mathbb{Z}$$



Lemma: F_n is small away from 0.

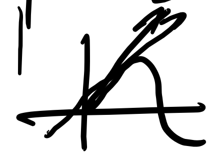
For $0 < \delta < 1$, $x \geq \delta$,

$$F_n(x) \leq \frac{1}{n} \frac{1}{(\sin \pi x)^2} \leq \frac{1}{n} \frac{1}{4x^2} \leq \frac{1}{4n\delta^2}$$

$$\lim_{n \to \infty} \left\{ \max_{\delta \leq |x| \leq 1} F_n(x) \right\} = 0.$$

$\cancel{\sin \pi x \leq \pi x}$

$$\frac{2x}{\pi} \leq \sin x \leq x.$$



$$\underline{\underline{f * F_n}} \to f \quad \text{in some sense}$$

$\sigma_n(f)$

$$\boxed{f * F_n - f} \to 0 \quad \text{in } \cancel{\text{some}} \text{ an appropriate sense.}$$

$$\int_0^1 f(x-t)\, \underline{\underline{F_n}}(t)\, dt - \int_0^1 f(x)\, F_n(t)\, dt$$

$$= \int_0^1 (f(x-t) - f(x))\, F_n(t)\, dt.$$

$$= \int_0^1 (f(x-t) - f(x))\, F_n(t)\, dt$$

$(f * F_n - f)(x)$

$$= \int_0^1 \frac{1}{2}\left[f(x+t) + f(x-t) - 2f(x)\right] F_n(t)\, dt.$$

Fundamental Theorem of Calculus :

$f \to$ Riemann integrable on $[0,1]$.

$$\boxed{F(a) = \int_0^a f(t)\, dt,} \quad \forall a \in [0,1]$$

$F'(x) = f(x) \quad \forall x \in [0,1].$

Defn: Suppose that $f \in L^1(S^1, m)$ and $x \in S^1$ such that $f(x) \neq \pm\infty$. Then x is called a **Lebesgue point** of f if

$$\boxed{\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0}$$

$$\left| \frac{\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt}{h} - f(x) \right| = \frac{1}{h} \left| \int_x^{x+h} (f(t) - f(x))\, dt \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt$$

If x is a Lebesgue point of f, then

$F(x) = \int_0^x f(t)\,dt$ is differentiable

and $F'(x) = f(x)$.

* If $f \in C(S^1)$, then every point of S^1 is a Lebesgue point for f.

<u>Proposition</u>: Let $f \in L^1(S^1, m)$. Almost every point of S^1 is a $\boxed{\text{Lebesgue point of } f}$. (Consequence of Lebesgue differentiation theorem).

Theorem (Fejér-Lebesgue).

If $f \in L^1(S^1, m)$ and x is a Lebesgue point of f. Then $\lim\limits_{n\to\infty} (f * F_n)(x) = f(x)$.

$\sigma_n(f)(x)$

Pf:

$$(f * F_n)(x) - f(x) = \int_0^1 \frac{1}{2}\left[f(x+t) + f(x-t) - 2f(x)\right] F_n(t)\,dt$$

To Prove: $\lim\limits_{n\to\infty} (f * F_n)(x) = f(x)$

Claim: $\lim\limits_{n\to\infty} \int_0^\delta \frac{1}{2}\left[f(x+t) + f(x-t) - 2f(x)\right] F_n(t)\,dt = 0$

Let $\varepsilon > 0$

$\exists\ 0 < \rho < \boxed{\delta}$ such that

$\int_0^t \Phi(t) < \boxed{\varepsilon t}$ for all $0 < t < \rho$.

$\Phi(t) := \int_0^t |f(x+u) + f(x-u) - 2f(x)|\, du \sim \boxed{o(t)}$

$\varphi(t) := [f(x+t) + f(x-t) - 2f(x)]$.

Let $n \in \mathbb{N}$ such that $n > \frac{1}{\rho^4}$.

Split the integral into the following three parts.

$I_1 := \frac{1}{2}\int_0^{1/n} \alpha(t)\, F_n(t)\, dt$

$\frac{1}{2}\int_0^{\delta} \alpha(t)\, F_n(t)\, dt$

$I_2 := \frac{1}{2}\int_{1/n}^{\rho} \alpha(t)\, F_n(t)\, dt$

$I_3 := \frac{1}{2}\int_{\rho}^{\delta} \alpha(t)\, F_n(t)\, dt$,

$\frac{2x}{\pi} \leq \sin x \leq x$

I_1: $I_1 \leq \frac{C}{n}\int_0^{1/n} \left(\frac{\sin n\pi x}{\sin \pi x}\right)^2 |\alpha(t)|\, dt$

$\leq \frac{C' n^2}{n}\int_0^{1/n} \frac{x^2}{x^2} |\alpha(t)|\, dt \leq C_1' \varepsilon$

I_2 : $I_2 \leq \frac{C_2}{n} \int_{1/n}^{\rho} \frac{|\varphi(t)|}{t^2}\, dt$ $\quad \left\| \left(\int \sin n\pi t \right)\left(\int \sin \pi t \right) \right\|_{\rho}^{2}$

$$= \frac{C_2}{n}\left(\frac{\Phi(\rho)}{\rho^2} - \frac{\Phi(1/n)}{1/n^2} + 2\int_{1/n} \frac{\Phi(t)}{t^2}\, dt \right)$$

$$< \frac{C_2}{n}\left(\frac{\varepsilon}{\rho} + 2\varepsilon \int_{1/n}^{\rho} \frac{1}{t^2}\, dt \right)$$

$\left(\frac{1}{n\rho} < \rho^3 < 1 \right.$

$$= \frac{C}{n}\left(\frac{\varepsilon}{\rho} + 2\varepsilon n - \frac{2\varepsilon}{\rho} \right)$$

$$< C\varepsilon$$

$\underline{\underline{I_3}}$: $|I_3| \le \dfrac{C_3}{n\rho^2} \int_\rho^\delta |\alpha(t)|\, dt$

$\left(n > \dfrac{1}{\rho^4} \right.$

$\sqrt{n} > \dfrac{1}{\rho^2}$

$\le \dfrac{C_3 \|\alpha\|_1}{n\rho^2}$

$\le \dfrac{C_3 \|\alpha\|_1}{\sqrt{n}}$

$|I_1 + I_2 + I_3| \le |I_1| + |I_2| + |I_3|$

$\left(C_1' + C_2' \right) \varepsilon + \dfrac{C_3 \|\alpha\|_1}{\sqrt{n}})$

Corollary. If $f \in L^1(S^1, m)$ and x is a point of continuity, then $\lim_{n\to\infty} \underbrace{\sigma_n(f)}_{f * F_n}(x) = f(x)$.

Corollary: If $f \in C(S^1)$, then $\forall\, x \in S^1$, $\lim_{n\to\infty} \sigma_n(f)(x) = f(x)$.

In other words, $\sigma_n(f)$ converges pointwise to f.

Corollary: $\hat{\ }: L^1(S^1, m) \to l_0(\mathbb{Z})$.

Let $f \in C(S^1)$, such that

$$\sum_{k=-\infty}^{\infty} |c_k| < \infty, \text{ i.e., } \hat{f} \in l^1(\mathbb{Z}) \cap l_0(\mathbb{Z}).$$

Then $S_n(f) \to f$ uniformly

$$S_n(f) = \sum_{k=-n}^{n} c_k(f) e_k$$

and $$f(x) = \sum_{k=-\infty}^{\infty} c_k(f) e_k(x)$$

Pf. $(c_k)_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$,

Clearly, $\boxed{S_n(f) \to g}$ uniformly.

$x_1, x, \dots, x_n$

$S_n(f) \to g$ pointwise.

$x_1, x_1, x, \dots, x, \dots x,$

pointw

$\Rightarrow \sigma_n(f) \to g$

...

$\Rightarrow \sigma_n(f) \to f$ a.e. pointw.

$\Rightarrow \boxed{f = g}$

Theorem: $\hat{\cdot} : L^1(S^1, m) \to c_0(\mathbb{Z})$

is injective.

Pf: If $\boxed{\hat{f} = \hat{g}}$, $\sigma_k(f) = \sigma_k(g)$ $\forall k \in \mathbb{Z}$

$\sigma_k(f) \longrightarrow f$ a.e.

$\quad \shortparallel$

$\sigma_k(g) \longrightarrow g$ a.e.

$$\boxed{f = g \text{ a.e.}}$$

Theorem : $\hat{\cdot} : C^\infty(S^1, \mathbb{R}) \to$ rapidly decreasing functions on $\mathbb{Z}$
$(\mathcal{S}(S^1))$ $\quad$ $(\underline{\underline{\mathcal{S}(\mathbb{Z})}})$

is a bijection.

Pf :- **Thm 7.1** (Rudin-1)

$\{f_n\} \to$ differentiable on $[a, b]$.

~~$\{f_n(x_0)\}$~~ $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$.

If $\{f_n'\}$ converges uniformly to g, then $\{f_n\}$ also converges uniformly to f, $g'(x) = \lim_{n \to \infty} f_n'(x) \ldots$

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty \qquad \sum_{k=-n}^{n} c_k e^{ikx}$$

$$\sum_{k\in\mathbb{Z}} |k|\,|c_k| < \infty$$

$$\sum_{k\in\mathbb{Z}} |k|^{\alpha} |c_k| < \infty, \quad \forall \alpha \geq 1.$$

$$f = \sum_{k\in\mathbb{Z}} \boxed{c_k}\, e_k$$

$$f' = (2\pi i)\sum_{k\in\mathbb{Z}} \boxed{k\, c_k}\, e_k, \quad f'' = \sum_{k\in\mathbb{Z}} (2\pi i)^2 k^2 c_k e_k$$