Analysis - 2

Main theme: Fourier analysis

What is Fourier analysis?

Qn: How can we write a given complex-valued $\boxed{\text{function}}$ as a sum (linear combination) of $\boxed{\text{elementary}}$ functions? $f : [a,b] \to \mathbb{C}$

Analogy with linear algebra: Let H be a self-adjoint linear transformation on $\boxed{\mathbb{C}^n}$, $f : \{1, \dots, n\} \to \mathbb{C}$

$e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ How do $\phi(H)$ act on $\mathbb{C}^n$?

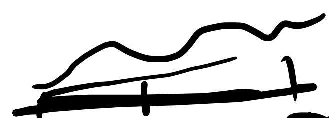
$x_1, \ldots, x_n$ is an orthonormal eigenbasis for H.

$Hx_1 = \lambda_1 x_1, \ldots \ldots$ $\quad Hx_n = \lambda_n x_n$.

$x = c_1 x_1 + \ldots + c_n x_n$ $\quad \boxed{(c_1, \ldots, c_n)}$

$Hx = \lambda_1 c_1 x_1 + \ldots + \lambda_n c_n x_n$ $\quad (\lambda_1 c_1, \ldots, \lambda_n c_n)$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$$ (heat equation in 1-D)



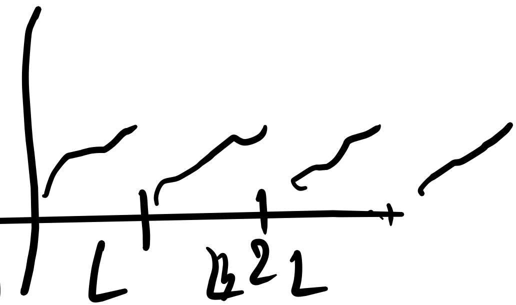
constant-coefficient PDE.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Periodic functions on $\mathbb{R}$

A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *periodic* with period $L>0$ if for every $x\in\mathbb{R}$,

$$f(x+L)=f(x).$$



Normalizing assumption:

If f is periodic with period L, then $F(x) := f(Lx)$ is periodic with period 1. and we will call F to be simply *periodic*.

The elementary functions are the <u>exponentials</u>.

For $k \in \mathbb{Z}$, let $e_k(x) := e^{2\pi i k x}$ $\forall x \in \mathbb{R}$.

e_k is a periodic function.

$$e_k(x+y) = e^{2\pi i k(x+y)} = e_k(x) \cdot \underline{\underline{e_k(y)}}$$

$$e_k(x+a) = \underbrace{C}_{e_k(a)}\, e_k(x)$$

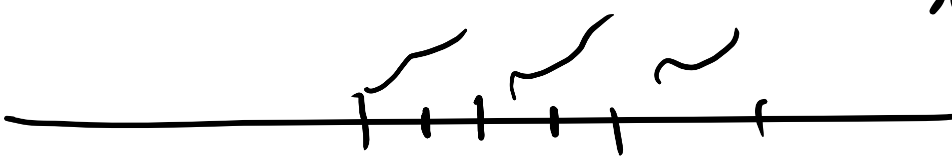
$$T_a \cdot e_k = c\, e_k$$

(Input of T_a is a function and output is its translate by a.)

$$\boxed{T_a \cdot e_k(x) = e_k(x+a)}$$

Any periodic function may be identified with a function on $\mathbb{R}/\mathbb{Z}$.

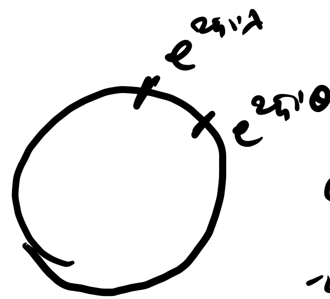
$x \sim y$ if $x - y \in \mathbb{Z}$.



$[x] \to$ equivalence class of x.

$\tilde{f}([x]) = f(x)$

$\mathbb{R}/\mathbb{Z} \sim S^1$



$d(e^{2\pi i\theta}, e^{2\pi i\lambda}) = |\theta - \lambda|$

S^1 has length 1.

If our function space is on a compact domain (interval) of $\mathbb{R}$ (Ω), and say it is $L^1(\Omega)$

<u>Remark</u>: We do not necessarily need the context of periodic functions to study Fourier analysis. If we are working on a compact domain, and our function space does not care about values on the boundary (i), then also such techniques may be used.

L^1-theory of Fourier transform on S^1

Let $f \in L^1(S^1, dm)$.

$$c_k(f) := \int_0^1 f(x)\, e^{-2\pi i k x}\, dx$$

$$= \int_{S^1} f(\theta)\, \theta^{-k}\, d\theta$$

are called the Fourier coefficients of f.

c_k is the k^{th} Fourier coefficient of f. $(k \in \mathbb{Z})$.

We have the following linear mapping.

$$\hat{\ } : L^1(S^1, dm) \longrightarrow \mathbb{C}^{\mathbb{Z}}$$

functions from $\mathbb{Z} \to \mathbb{C}$

$$\hat{f} \quad f \longrightarrow (\ldots\ldots, \; c_k(f) \ldots\ldots)$$

This linear mapping is called the Fourier transform.

Theorem 1 :

$\hat{\cdot} : L^1(S^1, m) \to \mathbb{C}^{\mathbb{Z}} \to \ell^\infty(\mathbb{Z}) \to C_b(\mathbb{Z})$

has range in $\ell^\infty(\mathbb{Z})$, and is a bounded linear transformation from $L^1(S^1, m)$ to $\ell^\infty(\mathbb{Z})$, with bound ≤ 1.

$T : V_1 \to V_2$

$\|Tx\| \le \|T\| \, \|x\|$

$$\boxed{\|\hat{f}\|_\infty \le \|f\|_1}$$

Pf. $|c_k(f)| = \left|\int_0^1 f(x) e^{-2\pi i k x} \, dx\right| \le \int_0^1 |f(x) e^{-2\pi i k x}| \, dx$

$\Rightarrow \sup_k |c_k(f)| \le \|f\|_1 \qquad = \|f\|_1$

Theorem 2: (Riemann-Lebesgue)

If $f \in L^1(S^1, m)$, then $c_n(f) \to 0$ as $|n| \to \infty$.

$c_0(f)$

Pf: $\chi_{[a,b]}$, $0 < a < b < 1$

$$c_k(\chi_{[a,b]}) = \int_0^1 \chi_{[a,b]} e^{-2\pi i k x}\, dx$$

$$= \int_a^b e^{-2\pi i k x}\, dx = \frac{1}{-2\pi i k}\left(e^{-2\pi i k b} - e^{-2\pi i k a}\right)$$

$$|c_k(\chi_{[a,b]})| \le \frac{1}{\pi k}$$

$$\lim_{|k| \to \infty} | c_k(\chi_{[a,b]}) | = 0.$$

Hence, if g is a step function on $[0,1]$,

then $\lim_{|k| \to \infty} | c_k(g) | = 0$.

Step functions are dense in $L^1(S^1)$.

$\exists$ g_ε step function s.t.

$$\int_0^1 | f(x) - g_\varepsilon(x) | \, dx < \varepsilon$$

$$|c_k(f)| = \left|\int_0^1 f(x)\, e^{-i2\pi k x}\, dx\right|$$

$$= \left|\int_0^1 (f(x) - g_\varepsilon(x))\, e^{-i2\pi k x}\, dx + \int_0^1 g_\varepsilon(x)\, e^{-2\pi i k x}\, dx\right|$$

$$\le \int_0^1 |f(x) - g_\varepsilon(x)|\, dx + \left|\int_0^1 g_\varepsilon(x)\, e^{-2\pi i k x}\, dx\right|$$

$$\le \varepsilon + \left|\int_0^1 g_\varepsilon(x)\, e^{-2\pi i k x}\, dx\right| \to 0$$

$$0 \le \limsup_{|k|\to\infty} |c_k(f)| \le \varepsilon$$

$$\limsup_{|k|\to\infty} |c_k(f)| = 0$$

$$\Rightarrow \lim_{|k|\to\infty} c_k(f) = 0$$

Corollary 3: Let $f \in C^{(k)}(S^1)$. Then

$$c_n(f) = o(n^{-k})$$

little o

that is, $$\lim_{|n| \to \infty} |n^k c_n(f)| = 0$$

Pf: $c_n(f) = \int_0^1 f(x) e^{-2\pi i n x}\, dx$

$$\int_0^1 f^{(1)}(x) e^{-2\pi i n x}\, dx = -\int_0^1 f(x)(-2\pi i n) e^{-2\pi i n x}\, dx$$

$$= (2\pi i n)\, c_n(f).$$

$$\boxed{\int_0^1 f^{(k)}(x)\, e^{-2\pi i n x}\, dx}$$ → continuous on S^1

$$= (2\pi i n)^k \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$$

$$= \boxed{(2\pi i n)^k\, c_n(f)}$$

$$\lim_{|n|\to\infty} \int_0^1 f^{(k)}(x)\, e^{-2\pi i n x}\, dx = ?\; 0$$

$$\Rightarrow \lim_{|n|\to\infty} (2\pi n)^k |c_n(f)| = 0$$

Mantra: Smoothness in f corresponds to decay properties in $\hat{f}$.

If $f \in C^\infty(S^1)$, then $\hat{f}$ ~~is~~ decays faster than reciprocal of any polynomial over $\mathbb{Z}$.

(This is actually bijection between $C^\infty(S^1)$ and rapidly decaying ~~de~~ sequences in $\ell^\infty(\mathbb{Z})$.

Apart from vector-space structure of $L^1(S^1, m)$, ~~...~~ it may also be endowed with a natural multiplicative structure.

Definition: $f, g \in L^1(S^1, m)$, $h = f * g$

$$h(x) = \int_{S^1} f(x-y)\, g(y)\, dy$$

$f(x-y) \cdot g(y)$ is a measurable function of x, y.

Use Fubini's theorem to conclude.

$$\|h\|_1 \le \int_{S^1} \int_{S^1} |f(x-y)| g(y) dy\, dx$$
$$= \int_{S^1} \int_{S^1} |f(x-y) g(y)| dx\, dy$$

$$\|h\|_1 \le \int_{S^1} |f(x-y)|\,dx \cdot \int_{S^1} |g(y)|\,dy$$

$$= \|f\|_1 \cdot \|g\|_1$$

If $f, g \in L^1(S^1, m)$, then $f * g \in L^1(S^1, m)$.

Properties of convolution:

(1) $\underline{f * g} = g * f$, $\int_{S^1} f(x-y)\,g(y)\,dy$

$= \int_{S^1} (f(y)\,g(x-y)\,dy$

$= g * f$.

(11) $(af_1 + f_2) * g = a f_1 * g + f_2 * g$

$L^1(S^1)$ is a complex algebra with $*$ as multiplication. (not a unital algebra.

S^1

$G \to$ finite abelian group.

$\mathbb{C}[G] = \sum_{g \in G} a_g \cdot g$

$\left(\sum_{g \in G} a_g \cdot g\right)\left(\sum_{h \in G} b_h \cdot h\right) = \sum_{g \in G} (a_{h_1} \cdot b_{h_2})\, g$

$h_1, h_2 \ldots h_1 h_2 = g$

$G \xrightarrow{\rho} Aut(V)$

$\sum a_g \cdot g \to \sum a_g \rho(g)$

Theorem 4: If $f \in L^p(S^1)$, $1 \le p \le \infty$, and $g \in L^1(S^1)$, then $h = f * g$ is well-defined and belongs to $L^p(S^1)$.

Moreover, $\|h\|_p \le \|f\|_p \cdot \|g\|_1$

$\|f * g\|_p$

$\|h_f - I\| > \frac{1}{2}$.

$e * f = f$

$g * $

$$L^1(S^1) \longrightarrow \mathcal{B}(L^p(S^1))$$

$$g \longrightarrow L_g(f) = g * f$$

Pf: Minkowski's integral inequality

$$\left(\int_X \boxed{\left(\int_Y |u(x,y)|\,dy\right)^p} dx\right)^{1/p} \le \int_Y \left(\int_X |u(x,y)|^p\,dx\right)^{1/p} dy$$

For us, $X = Y = S^1$, $\quad u(x,y) = f(x-y)\cdot g(y)$.

$$\underset{\hookrightarrow h}{\|f*g\|} = \left(\int_{S^1} |h(x)|^p\,dx\right)^{1/p} \le \int_{S^1}\left(\int_{S^1} |f(x-y)\,g(y)|^p\,dx\right)^{1/p} dy$$

$$= \left(\int_{S^1} |f(x-y)|^p\,dx\right)^{1/p} \cdot \left(\int_{S} |g(y)|\,dy\right)$$

$$\|f * g\|_p \le \|f\|_p \cdot \|g\|_1 .$$

Theorem 5: If $f, g \in L^1(S^1, m)$, then

$$c_n(f * g) = c_n(f) \cdot c_n(g) \quad \forall n \in \mathbb{Z}.$$

Pf:- $c_n(f * g) = \int_{S^1} \left(\int_{S^1} f(x-y) \cdot g(y)\, dy \right) e^{-2\pi i n x}\, dx$

$$= \int_{S^1} \left(\int_{S^1} f(x-y)\, g(y)\, e^{-2\pi i n x}\, dx \right) dy$$

$$= \int_{S^1}\int_{S^1} f(x-y)\, g(y)\, e^{-2\pi i n(x-y)} \cdot e^{-2\pi i n x}\, dx\, dy$$

$$= \left(\int_{S^1} f(x-y)\, e^{-2\pi i n(x-y)}\, dx\right)\left(\int_{S^1} g(y)\, e^{-2\pi i n y}\, dy\right)$$

$$= c_n(f) \cdot c_n(g)$$

~~Conttfr~~ given $\hat{f}$, how can we find f?

$$f \sim \sum_{k=-\infty}^{\infty} c_n(f) \cdot e^{inx}.$$