28/08/2021

Lecture 13

Previous class: Basic properties of harmonic functions.

$$\Delta u = 0 \qquad\qquad \nabla \cdot \nabla u = 0$$

$\Delta u = 0 \to$ potential

$\nabla \cdot \nabla u = 0 \to$ electric field

x_1

$$0 \longrightarrow \frac{1}{|x|} \qquad\qquad x_0 \longrightarrow \frac{1}{|x - x_0|}$$

u_1, u_2 are harmonic in D and continuously extend to ∂D.

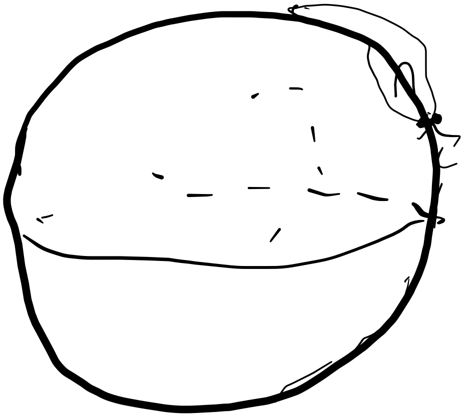
$$\frac{1}{|x-x_0|}$$

and $u_1 = u_2$ on ∂D.

Then, $u_1 \equiv u_2$ on $\overline{D}$

$\boxed{\Delta u = f}$ on ∂D.

Dirichlet problem for the unit sphere

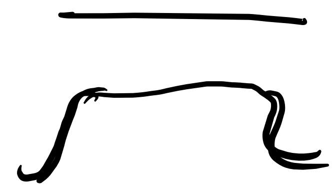


$$\boxed{\begin{aligned} u &= f \text{ on } S^{n-1} \\ \Delta u &= 0 \text{ in } \underline{\text{unit } B(0,1)}. \end{aligned}}$$

$L^p(S^{n-1})$

continuous function on S^{n-1}

Find such u.



Poisson kernel for the unit sphere

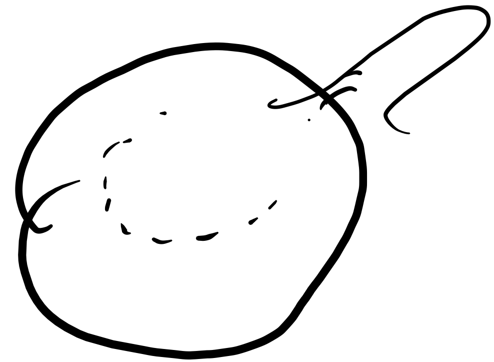
$\boxed{\Delta} u = f$

$$\boxed{\Delta u = \delta_0}$$

$$p(s,x) := \frac{1}{\omega_{n-1}} \frac{1-|x|^2}{|x-s|^{n}}$$

$r = |x| < 1$, $(x \in B(0,1))$

$s \in \Sigma_{n-1}$, $(|s| = 1)$



$$p(s,x) = \frac{1}{\omega_{n-1}} \frac{1-r^2}{(1-2r\cos\gamma + r^2)^{n/2}}$$

γ — angle between x, s

Theorem: If $\Sigma = \Sigma_{n-1} = \{ s \in \mathbb{R}^n : |s| = 1 \}$ denotes the surface of the unit sphere of $\mathbb{R}^n$, and ds the element of the surface area, then.

(a) $\phi(s, x) \geq 0$ $\forall s \in \Sigma$, and $|x| < 1$.

(b) $\int_{\Sigma} \phi(s, x)\, ds = 1$, whenever $|x| < 1$

$\to W(x)$

(c) For $x' \in \Sigma$, let $x = r x'$ $(0 \leq r < 1)$. Then for each $\delta > 0$,

$$\int_{s \in \Sigma,\ |s - x'| > \delta} \phi(s, r x')\, ds = \varepsilon_{\delta x} \to 0 \text{ as } r \to 1, \text{ uniformly in } x'.$$

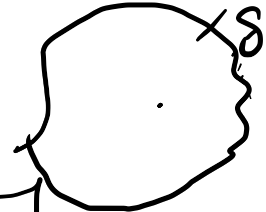
Dirichlet's Green's function

$$\Delta G(\boxed{x}, \boxed{x'}) = \delta(x - x')$$

w.r.t. x'

$x' \in \mathbb{R}^3$

$\boxed{x \in \partial D}$



$$G(x, x') = -\frac{1}{4\pi\, |x' - x|} + \frac{1}{4\pi\, |x'| \left| \frac{x'}{|x'|^2} - x \right|}$$

ω_2

$\boxed{\mathbb{R}^3 \times \mathbb{R}^3}$

$$z \to \frac{z}{|z|^2}$$

$$u(x) = \int_{B(0,1)} u(x')\, \delta(x-x')\, dx' \qquad \boxed{x \in B(0,1)}$$

$$= \int_{B(0,1)} u(x')\, \Delta G(x,x')\, dx'$$

$$-u(x) = \int_{B(0,1)} \left(G \cdot \Delta u\, dx' - \boxed{u\, \Delta G)\, dx'} \right.$$

$$= \int_{\partial B(0,1)} \left(G \frac{\partial u}{\partial n'} - u \frac{\partial G}{\partial n'} \right) ds'$$

$$\boxed{u(x) = \int_{\partial B(0,1)} u(s') \left(\frac{\partial G}{\partial n'}\right) ds'}$$

$\frac{\partial G}{\partial n'} \to$ Poisson kernel

$$\int_{\partial B(0,1)} \frac{\partial G}{\partial n'} ds' = 1$$

$$\int_{\partial B(0,1)} \underbrace{\Delta G}_{\delta(s-s')} \, ds' = 1$$

$$\int_{\partial B(0,1)} \frac{\partial G}{\partial n'} ds'$$

$$\underbrace{\Delta}_{\text{w.r.t } x}\left(\frac{\partial G}{\partial n'}\right) = \frac{\partial(\Delta G)}{\partial n'} = 0$$

Proof: (a) is obvious.

(c) $|s - x| = |s - r x'|$, $s, x' \in \Sigma$.

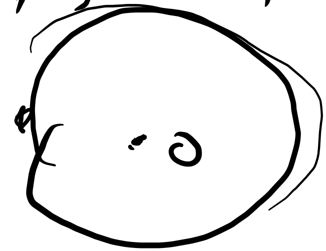
is bounded away from 0 (uniformly in x') when $s \in \Sigma$ satisfies $|s - x'| > \delta$.

$$\int_{\substack{\{|s-x'|>\delta\} \\ \cap \Sigma}} p(s, r x')\, d\sigma \to 0$$

as $r \to 1$,

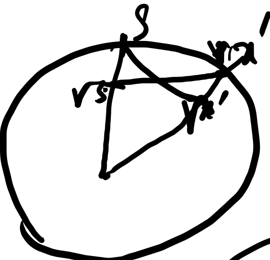
$$p(s, x) = \frac{1}{\omega_{n-1}} \frac{1 - |x|^2}{|x - s|^n} \leq \frac{1}{\delta^n}$$

(b) $p(x,s)$ is a harmonic function of x, $|x|<1$ for $s \in \Sigma$. By MVP,



$$\boxed{1 = \omega_{n-1}\, p(s,0)} = \omega_{n-1}\left(\frac{1}{\omega_{n-1}}\cdot\right)$$

$$= \frac{p(x)}{\omega_{n-1}}\, \omega_{n-1} \frac{1}{\omega_{n-1}} \int_\Sigma p(s, rx')\, dx' \quad r<1$$



$$= \int_\Sigma p(s, rx')\, dx', \text{ for } 0 \le r < 1$$

and $s, x' \in \Sigma$.

$|rx' - s| = |rs - x'| \Rightarrow p(s, rx') = p(x', rs)$

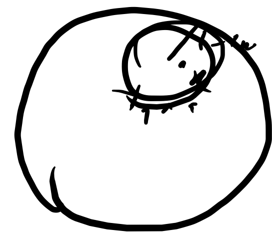
Thus $$\int_\Sigma p(x', s)\, dx' = 1$$

that is, $$\boxed{\int_\Sigma p(s, x)\, ds = 1} \quad \forall\, x \in \operatorname{int} B(0,1) \quad \blacksquare$$

<u>**Theorem**</u>: Suppose f is a continuous function on Σ, then the function u defined by

$$\underline{\underline{u(x)}} = \int_\Sigma f(s)\, \underline{\underline{p(s,x)}}\, ds \quad (|x|<1)$$

and $u(x) = f(x)$ $(|x| = 1)$, is harmonic for $\{x : |x| < 1\}$ and continuous on $\{x : |x| \le 1\}$.



Proof: **Part 1**, u has the MVP.

$r < 1 - |x|$,

$$M_{x,u}(r) = \frac{1}{\omega_{n-1}} \int_{\Sigma} \left(\int_{\Sigma} f(\xi)\, \phi(\xi, x + r\xi')\, d\xi \right) d\xi'$$

$$= \frac{1}{\omega_{n-1}} \int_{\Sigma} f(\xi) \left(\int_{\Sigma} \phi(\xi, x + r\xi')\, d\xi' \right) d\xi$$

(MVP for $p(s,x)$)

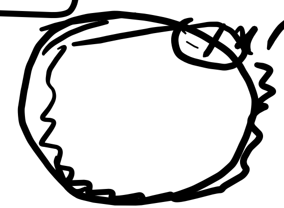
$$= \int_{\Sigma} f(s)\, p(s, x)\, ds$$

$$= u(x)$$

Thus u has MVP and is harmonic in the interior of the unit sphere.

<ins>Part 2</ins>: u extends continuously to $\overline{B(0,1)}$.

Let $x' \in \Sigma$, $0 \leq r < 1$, $\boxed{x = r x'}$ and



$$A_\delta := \{ s \in \Sigma : |s - x'| > \delta \}.$$

$$|u(x) - u(x')|$$

$$= \left| \int_{\Sigma} [f(s) - f(x')]\, p(s, rx')\, ds \right|$$

$$\le \int_{A_\delta} |f(s) - f(x')|\, p(s, rx')\, ds \quad (I_1)$$

$$+ \int_{\Sigma \setminus A_\delta} |f(s) - f(x')|\, p(s, rx')\, ds \quad (I_2)$$

$$I_2 \le \left\{ \sup_{|s-x'| \le \delta} |f(s) - f(x')| \right\} \cdot \int_{\Sigma} p(s, rx') \, ds$$

$$\le \left\{ \sup_{|s-x'| \le \delta} |f(s) - f(x')| \right\}.$$

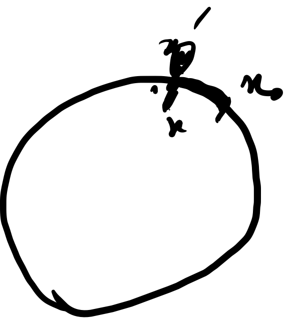
$$I_1 \le 2 \left\{ \sup_{t \in \Sigma} |f(t)| \right\} \int_{A_\delta} p(s, x) \, ds \to 0, \text{ as } r \to 1$$

(uniformly in x').

$\boxed{x_0} \in \Sigma$, $\quad x = r x'$, $x' \in \Sigma$.

$$|u(x_0) - u(x)| \le |u(x_0) - u(x')| + |u(x') - u(x)|$$

$$\le \underbrace{|f(x_0) - f(x')|}_{\to 0} + \underbrace{|u(x') - u(x)|}_{\to 0}$$



If x is sufficiently close to x_0,

$\Rightarrow$ $|x' - x|$, $1 - r$ are sufficiently close to 0.

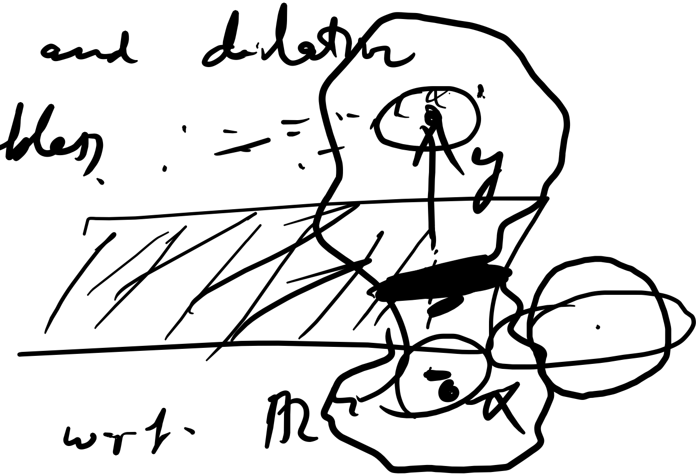
$\blacksquare$

Remark — The previous theorem solves the Dirichlet problem on the unit sphere (for f continuous).

Corollary — Let $D = \text{Int } B(x_0, a)$, f is a continuous function on ∂D.

Then
$$u(x) := \frac{1}{\omega_{n-1} a^{2-n}} \int_{\Sigma} f(x_0 + a s) \frac{a^2 - |x - x_0|^2}{|(x - x_0) - a s|^n} \, ds$$
(for $|x - x_0| < a$) and $u(x) = f(x)$ (for $|x - x_0| = a$) is harmonic D and continuous on $\bar{D}$.

Proof: appropriate translation and dilation (and change of variables)



Theorem (Reflection Principle)

Suppose $D \subseteq \mathbb{R}^{n+1}$ is a domain that is symmetric w.r.t. $\mathbb{R}^n$ (if $(x, y) = (x_1, \dots, x_n, y) \in D$, then $(x_1 \dots x_n, -y) \in D$).

If a continuous function u, defined on D satisfies $u(x, y) = -u(x, -y)$ ($u(x, -y) = -u(x, y)$) and is harmonic on the upper half-space

$\mathcal{D}^+ := \{(x,y)\in\mathcal{D} \,;\, y>0\}$

$(x,y) \hookrightarrow \mathbb{R}^2$

then it is harmonic on a nbd of $\mathcal{D}$.

Proof: suffices to show that u is harmonic on a nbd of $\mathcal{D}\cap\mathbb{R}^2$

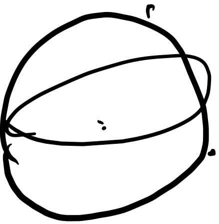
$= \{(x,y)\in\mathcal{D} \,;\, y=0\}$.

$(x,y) \hookrightarrow \mathbb{R}^2$

Define a harmonic function w on $\mathbb{R}^{n+1}$

on that ball. (using Poisson integral)

This integral vanishes when $\boxed{\int \ldots}$



$$\frac{u(x_0+as, at)}{a^2 - |(x-x_0, 0)|^2} = \frac{-u(x_0+as, -at)}{a^2-|(x-x_0,0)|^2}$$

$$\frac{a^2 - |(x-x_0, 0)|^2}{|((x-x_0-as), at)|^{n+1}} = \frac{a^2-|(x-x_0,0)|^2}{|(x-x_0-as, at)|^{n+1}}$$

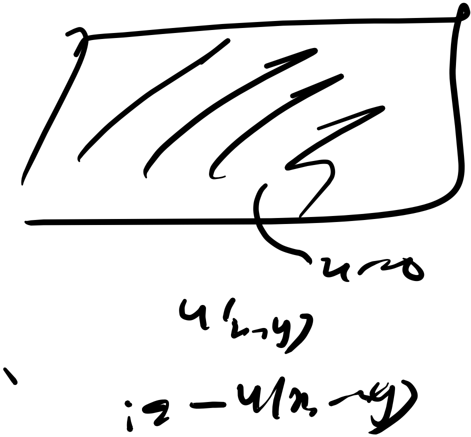
Corollary: Suppose u is continuous on $\overline{\mathbb{R}^{n+1}_+}$ (closed upper-half space) and is harmonic on open upper half space, and vanishes on $\partial\mathbb{R}^{n+1}_+$. If u is $\boxed{\text{bounded}}$ on the closed upper-half space, then u is identically 0.

Proof: Reflection principle

$\Rightarrow$ u can be extended to all of $\mathbb{R}^{n+1}$.

By Liouville theorem, u is constant,

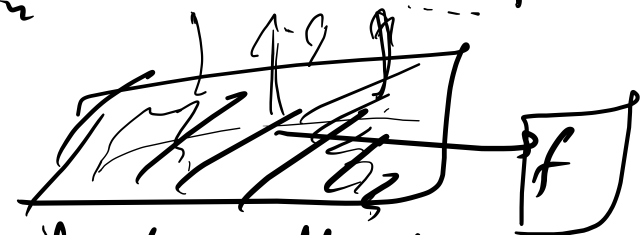
$\Rightarrow u \equiv 0$



Example. $u_1(x_1 \dots x_n, y) = \boxed{y}$.

$u_2(x_1 \dots x_n, y) = 0$

u_1, u_2 are harmonic function in $\mathbb{R}^{n+1}_+$, $L^2(S^{n-1})$

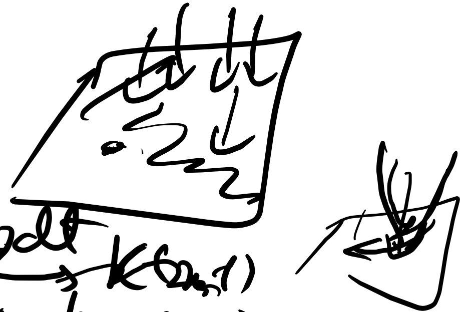


S^{n-1}

Poisson kernel for upper half space

$$P(t, \alpha) = \frac{\Gamma(n+1/2)}{\pi^{n+1/2}} \frac{\alpha}{(\alpha^2 + |t|^2)^{n+1/2}} = e^{-\log \alpha(\xi)}$$

If $f \in L^p(\mathbb{R}^n)$; $1 \le p \le \infty$



$$u(x,y) = \int_{\mathbb{R}^n} f(t)\, P(x-t, y)\, dt \qquad K(x,t)$$

(Poisson integral of f.) u is harmonic.

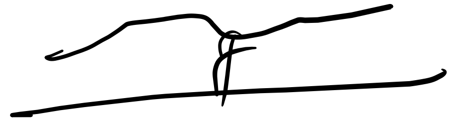
$$\lim_{y \to 0} u(x,y) = f(x) \quad \text{for almost every } x \in \mathbb{R}^n$$

$$u(\cdot, y) \to f \quad \text{in } L^p \text{ norm}$$

as $y \to 0$.

P → momentum

Q → position



$$\boxed{PQ - QP = i\hbar I}$$

$$(\operatorname{Var} f)(\operatorname{Var} \hat{f}) \geq C$$

spread of wave function of pos.

spread of wave function of mom.

$$\boxed{M_x \frac{d}{dx}} f - \frac{d}{dx} M_x f = i\hbar$$

$$i(\hbar)$$

$$\boxed{\frac{d^2}{dq^2}} e^{2\pi i p q} = -\boxed{(2\pi i p)}^2 e^{2\pi i p q}$$