24/08/21 Lecture-12

$$\langle \delta_0, \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$$\langle \partial^\alpha \delta_0, \varphi \rangle = (-1)^{|\alpha|} \langle \delta_0, \partial^\alpha \varphi \rangle$$

$$= (-1)^{|\alpha|} (\partial^\alpha \varphi)(0).$$

If $u \in \mathcal{S}'(\mathbb{R}^n)$ supported at $\{0\}$, then

$$u = \sum_{|\alpha| \le k} \boxed{a_\alpha} \partial^\alpha \delta_0 \text{, for some } k \in \mathbb{N} \cup \{0\}$$

and complex numbers a_α.

$\boxed{x^\alpha \partial^\beta \delta_0}$ supported at $\{0\}$.

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\Delta u = 0$, for $u \in \mathcal{S}'(\mathbb{R}^n)$

harmonic distribution

$(4\pi^2 |\xi|^2) \hat{u} = 0 \iff \operatorname{supp} \hat{u} \subseteq \{0\}$

$|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$

$\hat{u} = \sum_{|\alpha| \le n} a_\alpha \partial^\alpha \delta_0$

$\boxed{\Delta u = \delta_0}$

$\boxed{\Delta u = f}$

$\boxed{f} * \Delta u = f * \boxed{\delta_0}$

$= f$

$\Delta(f * u) = f$

$\boxed{L(u) = 0}$ (homogeneous differential eqn.)

$\boxed{L(u) = f}$

In order to solve $\boxed{\Delta u = f}$

first find v such that $\boxed{\Delta v = \delta_0}$

Then $f * v$

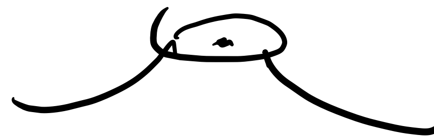
Proposition: For $n \geq 3$, we have

$$\Delta\left(|x|^{2-n}\right) = -(n-2)\,\omega_{n-1}\,\delta_0$$

surface area of S^{n-1}

and for $n = 2$,

$$\Delta\left(\boxed{\log|x|}\right) = 2\pi\,\delta_0,$$



$$\left(\Delta := \nabla^2 = \nabla\cdot\nabla\right)$$

For $n = 3$, $\quad \Delta\left(\frac{1}{|x|}\right) = -4\pi\,\delta_0$

Proof -

φ, ψ be scalar fields in $C^2(\mathbb{R}^n)$.

$$\nabla \cdot (\psi \nabla \varphi) = \psi \Delta \varphi + (\nabla \psi) \cdot (\nabla \varphi)$$

$$\nabla \cdot (\varphi \nabla \psi) = \varphi \Delta \psi + (\nabla \varphi) \cdot (\nabla \psi)$$

$$\boxed{\nabla \cdot (\psi \nabla \varphi - \varphi \nabla \psi) = \psi \Delta \varphi - \varphi \Delta \psi}$$

Green's second identity.

Δ ~~Outside of~~ $\{0\}$

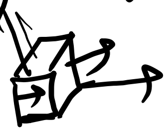
For $x_0 \in \mathbb{R}^n \setminus \{0\}$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$.

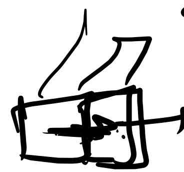
$\Delta |x|^{2-n}(x) = 0$.

Gauss' Divergence Theorem ; $\displaystyle\int_{\partial V} \vec{E} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{E})\, dV$

$(E_1, \dots, E_n)$

$\nabla \cdot E = \frac{\partial E_1}{\partial x_1} + \dots + \frac{\partial E_n}{\partial x_n}$





$$\int_\Omega \nabla\cdot\left(\underline{\underline{\varphi\nabla\psi - \psi\nabla\varphi}}\right)dV$$

$$= \int_{\boxed{\partial\Omega}} \left(\varphi\nabla\psi - \psi\nabla\varphi\right)\cdot d\vec{A}$$

$$\Omega := \mathbb{R}^n \setminus \overline{B(0,\varepsilon)}$$

$$\left[\int_\Omega \left(\varphi\Delta\psi - \boxed{\psi\Delta\varphi}\right)dV = \int_{\partial\Omega}\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)ds\right]$$

Take $\varphi = |x|^{2-n}$, $\psi = f \in C_c^\infty(\mathbb{R}^n)$.

$\Delta(|x|^{2-n}) = 0$, for $x \neq 0$,

$$\int_{|x|>\varepsilon} (\Delta f)(x) \boxed{|x|^{2-n}} \, dx$$

$$= - \int_{\partial B(0,\varepsilon)} \boxed{\varepsilon^{2-n}} \frac{\partial f}{\partial r} - f(\varepsilon\theta) \frac{\partial r^{2-n}}{\partial r} \, d\theta$$

surface area element.

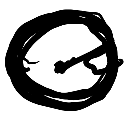
$$\left| \int_{\partial B(0,\varepsilon)} \boxed{\frac{\partial f}{\partial r}} d\theta \right| \leq \boxed{C_f} \, \varepsilon^{n-1}$$

$$\sup_{\theta} \left| \frac{\partial f}{\partial r}(\varepsilon\theta) \right|$$

and,

$$\int_{\partial B(0,\varepsilon)} \boxed{f(r(\theta))} \, \varepsilon^{1-n} \, d\theta \longrightarrow \omega_{n-1} f(0)$$

as $\varepsilon \to 0$



$$\lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} (\Delta f)(x) \, |x|^{2-n} \, dx = -(n-2) \, \omega_{n-1} f(0).$$

$$\int_{\mathbb{R}^n} (\Delta f)(x) \, |x|^{2-n} \, dx$$

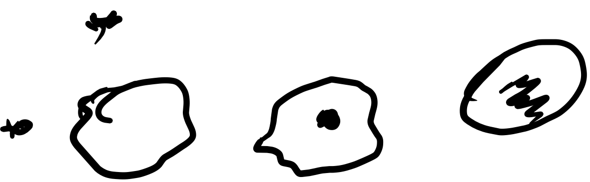
$$\langle \Delta |x|^{2-n}, f \rangle$$

$$\langle \Delta |x|^{2-n}, f \rangle = \langle -(n-2)\omega_{n-1}\delta_0, f \rangle$$

For $n \geq 3$,

$$\Delta |x|^{2-n} = -(n-2)\omega_{n-1}\delta_0$$

$$\Delta \varphi = 0 \iff \nabla \cdot \nabla \varphi = 0$$



$$\Delta\left(\frac{1}{|x|}\right) = 0$$ in dim 3.

$\boxed{\Delta u = 0}$ on $\mathbb{R}^n$

$\Rightarrow$ u is a harmonic polynomial.

$$\boxed{\Delta u = -(n-2)\,\omega_{n-1}\,\delta_0}, \quad u = \begin{cases} \dfrac{1}{|x|^{n-2}}, & n \geq 3 \\ \log|x|, & n = 2 \end{cases}$$

"fundamental solution":

$\Delta u = 0$ on $\mathbb{R}^n - \{0\}$

(Laplace's equation on the punctured Euclidean space).

$$\boxed{L(u) = \delta_0}$$

$$L(u) = f$$

∇ α: potential $\rightarrow$ $\boxed{\frac{1}{|x|}}$ $\boxed{n=3,}$

$\nabla \alpha$ $\rightarrow$ electric field

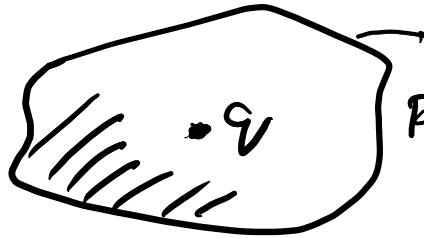
$\boxed{\Delta \alpha}$ $\rightarrow$ divergence of electric field

Outside of the region where there is electric charge, the electric field is divergence free. (Lyapunov law)

$$\boxed{\Delta \alpha = 0}$$ in the region where there is no charge.

$$\boxed{\Delta\varphi = C\delta_0}$$

$$\boxed{\Delta\varphi = \delta_0}$$



$$\text{Flux} = \frac{q}{\boxed{\varepsilon_0}}$$

$$\Delta\varphi_1 = \delta_0$$

$$\Delta\varphi_2 = \delta_0$$

$$\Delta(\varphi_1 - \varphi_2) = 0 \text{ on } \mathbb{R}^3$$

$$\boxed{a_0}\frac{}{|x-x_0|} + \frac{a_1}{|x-x_1|} \text{ on } \mathbb{R}^3 - \{x_0, x_1\}$$

$$\forall a_0, a_1 \in \mathbb{R}.$$

$$\frac{a_0}{|x - x_0|} \quad \boxed{\sum_{i=1}^{n} \frac{a_i}{|x - x_i|}}$$

$$\left(\sum_{i=1}^{n} a_i = 1 \right)$$

$\underline{\Delta u = 0}$ (outside of C
or outside of surface

and $\Delta u = f$ on C, resp. S)
respectively.

$$\boxed{\mathbb{C} \cong \mathbb{R}^2}$$

Basic Properties of Harmonic Functions

Defn. A function, u, defined in a domain D (open connected subset of $\mathbb{R}^n$) is said to be **harmonic** if $u \in C^2(D)$ and satisfies Laplace's equation.

$$\boxed{\Delta u = \sum_{i=1}^{n} \boxed{\frac{\partial^2 u}{\partial x_i^2}} = 0} \to \nabla \cdot (\nabla u)$$

superharmonic fn: $\boxed{\Delta u \leq 0}$

[subharmonic] $\boxed{\Delta u \geq 0}$

__Examples__:

(1) $u(x) = \begin{cases} \dfrac{1}{|x|^{n-2}}, & n \geq 3 \\ \log |x|, & n = 2 \end{cases}$

is harmonic on $D = \mathbb{R}^n - \{0\}$.

(2) [If f is holomorphic on $D \subseteq \mathbb{C} \cong \mathbb{R}^2$, then $\underbrace{\operatorname{Re} f}_{u}$, $\underbrace{\operatorname{Im} f}_{v}$ are harmonic on D.]

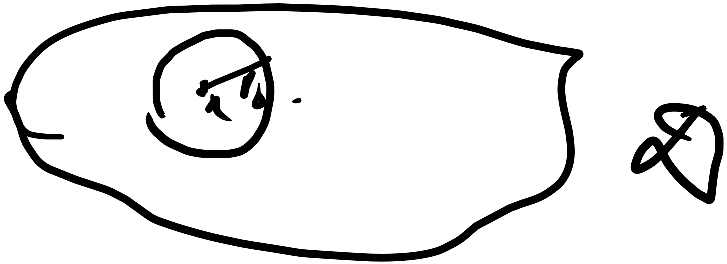
Mean value of a function, u, taken over the surface of a sphere of radius $r>0$ about the point x is defined by..

$$M_{x,u}(r) = \frac{1}{\omega_{n-1}} \int_{\Sigma_{n-1}} u(x+r\xi')\, d\xi'$$

Theorem: (Mean Value Property of Harmonic Functions)
If u is harmonic in a domain D and if the sphere of radius r_0 about $x \in D$ is contained in D, then for $0 \le r \le r_0$

$$\boxed{u(x) = M_{x,u}(r)}$$

Proof:



Green's identities: (First identity)

$$\nabla\cdot(\psi\nabla\varphi) = \psi\Delta\varphi + \nabla\psi\cdot\nabla\varphi$$

(Second identity)

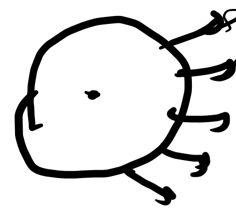
$$\boxed{\nabla\cdot(\varphi\nabla\psi - \psi\nabla\varphi) = \varphi\Delta\psi - \psi\Delta\varphi}$$

Step I :

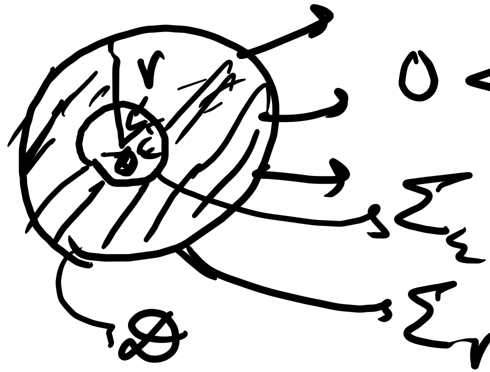
$$0 = \int_{B(x,r)} \Delta u(y)\,dy$$

$$= \int_{B(x,r)} \nabla\cdot(\nabla u)(y)\,dy$$

$$= \int_{\partial B(x,r)} \boxed{\frac{\partial u}{\partial \nu}}\,dy$$



Step II:



$0 < \varepsilon < r \le r_0$.

(WLOG, assume that $0 \in D$.)

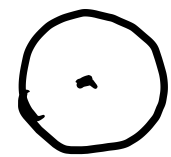
Apply Green's theorem to the function u and,

$$v(x) = \begin{cases} \dfrac{1}{|x|^{n-2}}, & \text{if } n \ge 3 \\ \log |x|, & \text{if } n = 2 \end{cases}$$

$$0 = \int_D \underline{\left(v \overset{0}{\cancel{\Delta u}} - u \overset{0}{\cancel{\Delta v}}\right)} \, dy$$

$$= \int_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$ (Gauss's divergence theorem)

$$= \int_{\boxed{\partial B(x,r)}} \left(v \overset{\text{integral } 0}{\cancel{\frac{\partial u}{\partial n}}} - u \frac{\partial v}{\partial n} \right) ds$$



$$- \int_{\partial B(x,\varepsilon)} \left(v \overset{\text{integral } 0}{\cancel{\frac{\partial u}{\partial n}}} - u \frac{\partial v}{\partial n} \right) ds$$

($\because$ v is constant on $\boxed{\partial B(x,r)}$ $\boxed{\partial B(x,\varepsilon)}$ respectively)

$$= \int_{\partial B(x,\varepsilon)} u \frac{\partial v}{\partial n}\, ds - \int_{\partial B(x,r)} u \frac{\partial v}{\partial n}\, ds \; (=0)$$

For $0 < \varepsilon < r \leq r_0$,

$$\int_{\partial B(x,\varepsilon)} u \boxed{\frac{\partial v}{\partial n}}\, ds = \int_{\partial B(x,r)} u \frac{\partial v}{\partial n}\, ds$$

$$\frac{1}{r^{2-n}} \quad \to \quad -(n-2)\frac{1}{|x|^{n-1}}$$

$$\frac{d}{dr} r^{2-n} = (2-n) r^{1-n}$$

$$\int_{\partial B(x,\varepsilon)} u \, -(n-2) \frac{1}{\varepsilon^{n-1}} \, dy$$

$$= \int_{\partial B(x,r)} u \left(-(n-2) \frac{1}{r^{n-1}} \right) dy$$

$$\Rightarrow \boxed{\underbrace{\frac{1}{\omega_{n-1} \varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u \, dy}_{M_{x,u}(\varepsilon)} = \underbrace{\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(x,r)} u \, dy}_{M_{x,u}(r)}}$$

$$M_{x,u}(\varepsilon) = M_{x,u}(r)$$

$$\to u(x)$$

$$\forall\, 0 < \varepsilon < r \leq R_0$$

$$\lim_{\varepsilon \to 0} M_{x,u}(\varepsilon) = u(x)$$

Thus, $\boxed{M_{x,u}(r) = u(x)}$ $\forall\ r \leq r_0$

∎

<u>Corollary</u>: (Maximum Principle for Harmonic Function)

Suppose a real-valued harmonic function u, defined on a domain D satisfies

$$A = \sup_{x \in D} u(x) < \infty, \text{ then}$$

$u(x) < A$ for all $x \in \mathcal{D}$, provided u is not a constant function.

(in other words, suppremum is not attained in $\mathcal{D}$).

Proof) - Suppose $u(x) = A$, for some $x \in \mathcal{D}$.

$\underline{\underline{M_{x,u}(r)}} = A$, for all r such that $B(x,r) \subseteq \mathcal{D}$
$u(x)$

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(x,r)} u(y)\, dy = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(x,r)} A\, dy$$

$(= A)$

For all $y \in B(x, r)$, we have

$$u(y) = A$$

In summary, if $u(x) = A$, there is a nbd of x in D where u is identically equal to A.

Hence, $u^{-1}(\{A\})$ is open.

$u^{-1}(\{A\})$ is closed as u is continuous.

Since D is connected, we have

$$u^{-1}(\{d\}) = D$$

or u is a constant function.

$\boxed{S}$

Looking at $-u$ we have the minimum principle for harmonic functions.

(that is, minimum of u is not attained in D.)

Corollary: If u is continuous on the closure $\overline{D}$ of the **bounded** domain D and is harmonic on D, then the maximum (minimum) of u is attained **only** on the boundary $\partial D = \overline{D} - D$ of D, provided u is not a constant function.

~~Proof~~. Remark: Harmonic functions (with continuous extension to $\overline{D}$) are governed by their boundary behaviour.

u_1, u_2 s.t. are harmonic on D

and $u_1 = u_2$ on ∂D.

$u_1 - u_2$ is harmonic on D

and $u_1 - u_2 = 0$ on ∂D.

By max. principle, $\max_{x \in \bar{D}} u_1 - u_2 = 0$

" min. $\min_{x \in \bar{D}} u_1 - u_2 = 0$

$\Rightarrow \boxed{u_1 \equiv u_2}$ on $\bar{D}$.

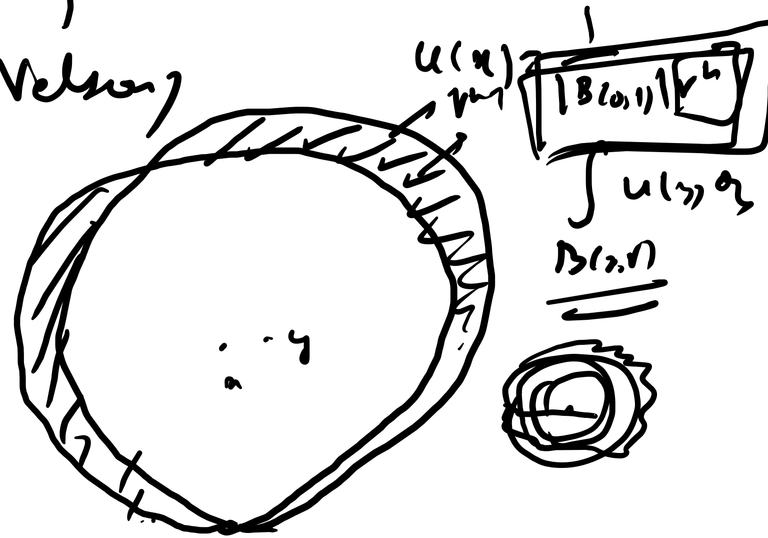
Corollary: (Liouville's theorem)

If u is harmonic on $\mathbb{R}^n$ and bounded, then u is a constant function.

Proof (due to Edward Nelson)

Pick any $x, y \in \mathbb{R}^n$.

To show: $u(x) = u(y)$.

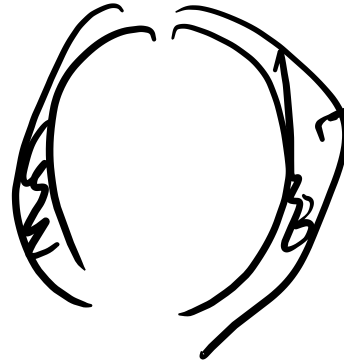


$u(x) = M'_{x,u}(r)$

$u(y) = M_{y,u}(r)$

$u(x) - u(y) = M'_{x,u}(r) - M'_{y,u}(r)$

As $r \to \infty$,



$|B(x,r)|$ grows as r^n,

$u(x) - u(y) \longrightarrow 0$ as $r \to \infty$

$\Rightarrow \boxed{u(x) = u(y)}$

Lemma: Suppose u is twice differentiable in D (in $C^2(D)$) and $M_{x,u}(r) = u(x)$ whenever $B_{x,u}(x, r)$ is in D, then u is harmonic in D.

Proof: Let $x \in D$.

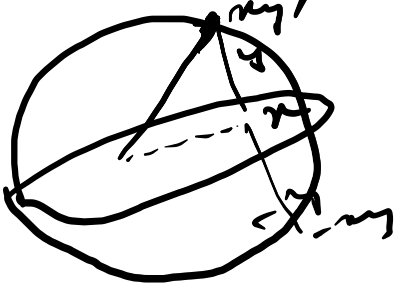
$M_{x,u}(r)$ is twice differentiable with respect to r.

$$M''(r) = \frac{1}{n} \Delta u(x).$$

$$M_{x,u}(r) = \frac{1}{\omega_{n-1}} \int_{\Sigma_{n-1}} u(x+rt')\,dt'$$

$$\boxed{\frac{d^2}{dr^2} M_{x,u}} = \frac{1}{\omega_{n-1}} \int_{\Sigma_{n-1}} \left(\sum_{j,k=1}^{n} u_{jk}(x+rt')\, t_j' t_k' \right) dt'$$

$$M''_{x,u}(0) = \frac{1}{\omega_{n-1}} \sum_{j,k=1}^{n} \underbrace{\left(\int_{\Sigma_{n-1}} t_j' t_k' \, dt' \right)}_{\frac{\omega_{n-1}}{n}\delta_{jk}} u_{jk}(x)$$



For $j \neq k$, $$\underbrace{\int_{\Sigma_{n-1}} \boxed{t_j'^2}\, dt'}_{S} = \int_{\Sigma_{n-1}} t_k'^2\, dt'$$

$$n S = \int_{\Sigma_{n-1}} \underbrace{\left(\sum_{j=1}^{n} t_j'^2 \right)}_{1} dt' = \omega_{n-1}$$

$$\Rightarrow S = \frac{\omega_{n-1}}{n}$$

Hence,

$$M_{x,u}''(0) = \frac{1}{n} \sum_{j=1}^{n} u_{jj}(x)$$

$$= \frac{1}{n} \Delta u(x)$$

Since, $\underline{\underline{M_{x,u}(r) = u(x)}}$, $\forall\, r \le r_0$ (s.t. $B(x,r) \subseteq D$)

$$M''_{x,u}(0) = 0$$

$$= \frac{1}{n} \Delta u(x)$$

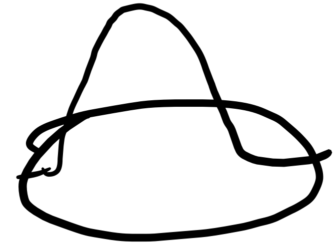
$\Leftrightarrow \Delta u = 0$, $\forall\, x \in D$.

$\Leftrightarrow$ u is harmonic

Theorem :- Suppose u is a continuous function on D satisfying MVP(mean value property). ; then u is harmonic and has continuous partial derivatives of all orders in D.

Proof :- Assume D is a sphere.

Extend u to $\mathbb{R}^n$ by defining $u=0$ outside of $\overline{D}$, Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a smooth radial function such that $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$



and $\operatorname{supp}(\varphi) \subseteq B(0,1)$...

$$\boxed{\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)}$$

Using polar coordinates..

$$\boxed{u_\varepsilon}(x) := u * \varphi_\varepsilon$$

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(t)\, u(x-t)\, dt$$

$$= \int_0^\varepsilon \varphi_\varepsilon(r) \left(\int_{\Sigma_{n-1}} u(x - r\ell)\, d\ell \right) r^{n-1}\, dr$$

$$= \int_0^\varepsilon \varphi_\varepsilon(r)\, \omega_{n-1} \underline{\underline{M_{x,u}(r)}}\, r^{n-1}\, dr$$

If ε is less than $d(x, \partial\Omega)$, then

$$\boxed{M_{x,u}(r) = u(x)} \qquad 0 \le r \le \varepsilon.$$

$$u_\varepsilon(x) = u(x) \left[\underbrace{\omega_{n-1} \int_0^\varepsilon \varphi_\varepsilon(r) \cdot \cancel{\omega_{n}}\, r^{n-1}\, dr}_{\int_{B(0,\varepsilon)} \varphi_\varepsilon(x)\, dx} \right]$$

$=$

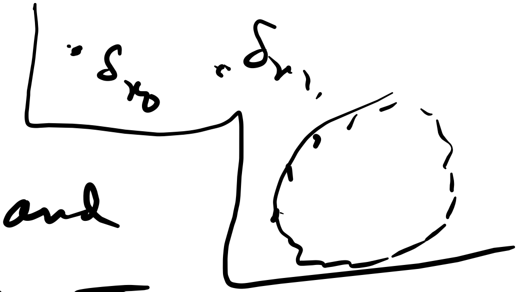
Hence, u is smooth.

Corollary:- Suppose $\{u_n\}$ is a sequence of harmonic functions defined on D. If this sequence converges uniformly to a function u on each compact subset of D, then u is also harmonic.

Proof:- u is continuous and satisfies MVP.

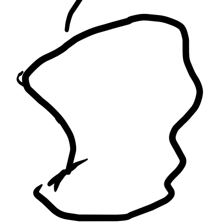
(This finishes our discussion on basic properties of harmonic functions)

The Dirichlet problem:



Suppose D is a region with compact closure $\overline{D}$, and $\boxed{f}$ is continuous on $\partial D = \overline{D} \setminus D$.

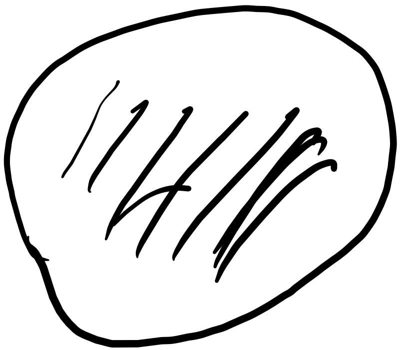
(continuous charge distribution) f



Does there exist a continuous function u on $\overline{D}$ such that:

(1) u is harmonic when restricted to D,

(2) $u(x) = f(x)$ when $x \in \partial D$.



$\mathbb{R}^4$

hyperspace of $\mathbb{R}^{n+1}$

$$\Delta u = \delta_{x=0}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right) dx$$

$$0 \quad \infty$$

$$x \to x - \frac{1}{x}$$

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right) dx \qquad \int_{-\infty}^{\infty} f\left(\frac{1}{x}-x\right) dx$$

$$\boxed{\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\,dx} \qquad \infty$$

$$\int_{-\infty} f\left(\frac{1}{x}-x\right)\frac{1}{x^2}\,dx$$

$$x \quad \frac{1}{x}$$

$$\int_0^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx$$

$x \to \frac{1}{x}$

$$\boxed{x} \to \boxed{x-\frac{1}{x}}$$

$$= \int_{-\infty}^{\infty} f\left(\frac{1}{x}-x\right)\left(1+\frac{1}{x^2}\right)dx$$

$$\int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx$$

$$x \to x-\frac{1}{x}$$

$$\int_0^{\infty} f\left(x-\frac{1}{x}\right)\frac{(1+x^2)}{x^2}\,dx$$

$$\int f\left(\frac{1}{x}-x\right)(1+x^2)$$