21/08/2021 **Lecture-11**

$f$ $\quad \boxed{L_f} \longrightarrow \boxed{\varphi} \longrightarrow \boxed{\int_{\mathbb{R}^n} f \varphi \, dx}$

test function

Weierstrass function - continuous fn on $\mathbb{R}$ which is nowhere differentiable

**Examples of tempered distribution:**

(1) Dirac mass at origin $\delta_0$. $\langle \delta_0, f \rangle = f(0)$

(2) $\boxed{L^1_{loc}(\mathbb{R}^n)} \subseteq \boxed{\mathcal{D}'(\mathbb{R}^n)}$, $L^p(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$?

$1 \le p \le \infty$

(3) Any finite Borel measure is a tempered distribution.

$$\langle \boxed{\mu}, \varphi \rangle = \boxed{\int_{\mathbb{R}^n} \varphi(x)\, d\underline{\underline{\mu}}(x)}$$

(4) Any tempered measure on $\mathbb{R}^n$ is a tempered distribution.

$m \to$ Lebesgue measure (tempered distribution)

$$m(B(0, r)) = C_n \boxed{r^n}$$

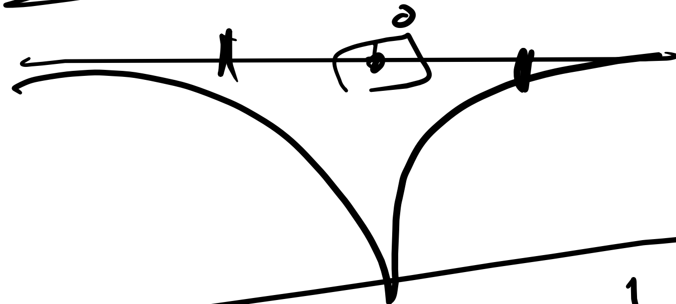
$r > 0$.

$\mu(\underline{\underline{B(0,r)}}) \leq C_n (1+r)^{n}$

$\forall r > 0$

$m$ $\boxed{\mu}(E) = \int_E \boxed{f}\, dm$

(5) $\log|x|$ is a tempered distribution.



$$\int_{\mathbb{R}^n} \log|x| \cdot \boxed{\varphi}\, dx$$

$$\boxed{|\log r| \le \frac{1}{r^m}} \quad \forall\, m \in \mathbb{N},$$

$$|\log \varepsilon| < \frac{1}{\varepsilon^k} \qquad \boxed{\log r < r^{\boxed{k}}} \text{ for } \boxed{r >> 1}$$

$$\log r^{1/m} < 1 \qquad \left|\log \frac{1}{r}\right| < \qquad \varepsilon = \frac{1}{r}$$

$$\left|\log \frac{1}{\varepsilon^c}\right| < 1$$

In 1 dimension eg

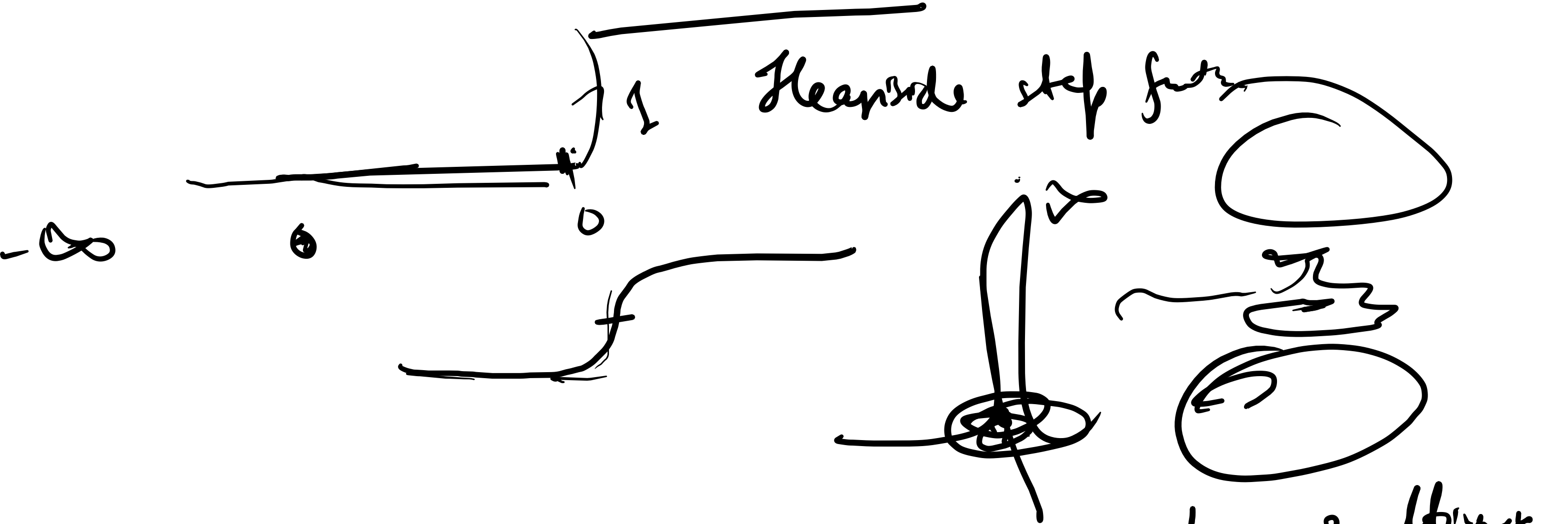
$$\frac{1}{r^{1-\delta}} \qquad \delta > 0$$

is integrable

$$\frac{1}{|x|^{n-\delta}} \text{ is integrable}$$

Theme → Think of "distributions" or "tempered distributions" as "generalized functions".

We want to implement "nice" operations like differentiation, convolution, Fourier transform on these generalized functions.




**Defn**: Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\alpha$ be a multi-index.
The $\alpha^{th}$ derivative of $u$ is defined by.

$$\boxed{\langle \boxed{\partial^\alpha u}, f \rangle} := \boxed{(-1)^{|\alpha|} \langle u, \boxed{\partial^\alpha f} \rangle} \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} \boxed{\partial^\alpha \varphi} \, \boxed{\psi} \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \boxed{\varphi} \cdot \partial^\alpha \psi \, dx \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

$f_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$

$\Rightarrow \quad \partial^\alpha f_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$

$\Rightarrow \langle u, \partial^\alpha f_k \rangle \to 0$ in $\mathbb{C}$

**Defn**: Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Fourier transform of $u$ (denoted by $\hat{u}$) is defined by:

$$\langle \hat{u}, f \rangle := \boxed{\langle u, \hat{f} \rangle}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

$\left( \int_{\mathbb{R}^n} \boxed{\hat{f}}\, g \,dx = \int_{\mathbb{R}^n} f \hat{g} \,dx \right)$

Inverse Fourier transform of $u$ (denoted $u^\vee$) is defined by:

$$\langle \boxed{u^\vee}, f \rangle := \langle u, f^\vee \rangle, \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

**Example:** (1) $\hat{\delta}_0 \equiv 1$ ↳ constant function.

$$\langle \hat{\delta}_0, \varphi \rangle := \langle \delta_0, \hat{\varphi} \rangle$$

$$= \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\, dx$$

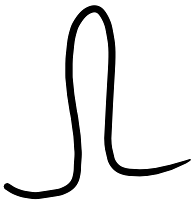
$$= \int_{\mathbb{R}^n} 1\, \varphi(x)\, dx = \langle 1, \varphi \rangle$$

$$\hat{1} = \delta_0$$

position (highly localized)

(ii) $\langle \boxed{\partial^\alpha \delta_0}, f\rangle = (-1)^{|\alpha|}\langle \delta_0, \partial^\alpha f\rangle$

$= (-1)^{|\alpha|} (\partial^\alpha f)(0)$.



* Any "function" ($L^1_{loc}$ with growth condition) is differentiable.

(iii) $\boxed{(\partial^\alpha \delta_0)^\wedge = (2\pi i \xi)^\alpha}$

$\langle (\partial^\alpha \delta_0)^\wedge, \varphi\rangle = (-1)^{|\alpha|}\langle \delta_0, \partial^\alpha \hat{\varphi}\rangle$

$$= (-1)^{|\alpha|} \langle \delta_0, \left( (-2\pi i x)^\alpha \varphi(x) \right)^\wedge \rangle$$

$$= (-1)^{|\alpha|} \left( (-2\pi i x)^\alpha \varphi(x) \right)^\wedge (0)$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-2\pi i x)^\alpha \varphi(x)\, dx$$

$$= \int_{\mathbb{R}^n} \underbrace{(2\pi i x)^\alpha}_{\text{polynomial funct.}} \varphi(x)\, dx = \langle (2\pi i x)^\alpha, \varphi \rangle$$

$$(\mathrm{W})\ \langle \delta_{x_0}, \varphi \rangle := \varphi(x_0), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$$\langle \boxed{\hat{\delta}_{x_0}}, \varphi \rangle = \langle \delta_{x_0}, \hat{\varphi} \rangle$$

$$= \hat{\varphi}(x_0)$$

$$= \int_{\mathbb{R}^n} \varphi(x) \boxed{e^{-2\pi i \langle x, x_0 \rangle}} \, dx$$

$$= \langle \boxed{e^{-2\pi i \langle x, x_0 \rangle}}, \varphi \rangle$$

$$\boxed{\hat{\delta}_{x_0} = e^{-2\pi i \langle x, x_0 \rangle}}$$

(V) $e^{|x|^2}$ is not a tempered distribution.

$\boxed{(1+|x|^2 \ldots}$ $\frac{\ldots |x|^{2k}}{k!}$

The Fourier transform of any locally integrable function with polynomial growth at infinity is defined as a tempered distribution.

For $f, g \in \mathcal{S}(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, $a > 0$,

(1) $$\int_{\mathbb{R}^n} g(x) f(x-t)\, dx = \int_{\mathbb{R}^n} g(x+t) f(x)\, dx$$

$$\langle g, \tau_t f \rangle = \langle \tau_{-t} g, f \rangle$$

(2) $\int_{\mathbb{R}^n} g(ax) f(x)\, dx$

$$= \int_{\mathbb{R}^n} g(x)\, a^{-n} f\left(\frac{x}{a}\right) dx$$

$$\left\langle \underline{\underline{\delta_a g}}, f \right\rangle = \left\langle g, a^{-n} \delta_{1/a}(f) \right\rangle$$

(3) $\int_{\mathbb{R}^n} \tilde{g}(x) f(x)\, dx = \int_{\mathbb{R}^n} g(x) \tilde{f}(x)\, dx$

$$\langle \tilde{g}, f \rangle = \langle g, \tilde{f} \rangle .$$

**Def^n:** The translation $\tau_t(u)$, the dilation $\delta_a(u)$, and the reflection $\tilde{u}$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ are defined as follows:

(1) $\langle \tau_t(u), f \rangle = \langle u, \tau_{-t}(f) \rangle$, $\forall f \in \mathcal{S}(\mathbb{R}^n)$.

(2) $\langle \delta_a(u), f \rangle = \langle u, \frac{1}{a^n}\delta_{1/a}(f) \rangle$, $\forall f \in \mathcal{S}(\mathbb{R}^n)$

(3) $\langle \tilde{u}, f \rangle = \langle u, \tilde{f} \rangle$, $\forall f \in \mathcal{S}(\mathbb{R}^n)$.

$\boxed{L(w) = f}$ , $\boxed{L(w) = \delta_0}$

constant coefficient differential operator

$f * \boxed{L}(w) = L(f * w)$

$f * \delta_0 = f$

If $\boxed{L(u) = \delta_0}$, then (roughly speaking) $\boxed{L(f * u) = f}$

For $f, g, h \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \underline{\underline{(h * g)(x)}}\, f(x)\, dx \quad \left( \langle h * g, f \rangle \right)$$

$$= \int_{\mathbb{R}^n} g(x) (\tilde{h} * f)(x)\, dx$$

$$\langle g, \tilde{h} * f \rangle$$

**Defn**: Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $h \in \mathcal{S}(\mathbb{R}^n)$. The convolution is defined by:

$$\langle \underline{\underline{h * u}}, f \rangle = \langle u, \underline{\tilde{h} * f} \rangle, \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

<u>Example</u>: Let $u = \delta_{x_0}$, $f \in \mathcal{S}(\mathbb{R}^n)$.

$$f * \delta_{x_0} = f(x - x_0).$$

$$\left\langle \cancel{f * \delta_{x_0}}\; f * \delta_{x_0}, h \right\rangle$$

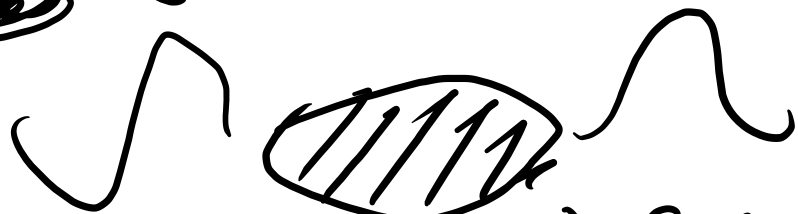
$$= \left\langle \delta_{x_0}, \tilde{f} * h \right\rangle = (\tilde{f} * h)(x_0)$$

$$= \int_{\mathbb{R}^n} \boxed{f(x - x_0)}\, h(x)\, dx$$

$$= \left\langle \boxed{\tau_{x_0} f}, h \right\rangle$$

Defn. (support of a distribution)

$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$



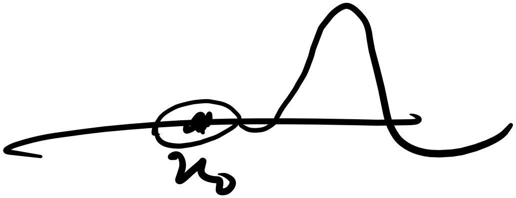

For $u \in \mathcal{D}'(\mathbb{R}^n)$ $(\supseteq \mathcal{S}'(\mathbb{R}^n))$

$\int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx = 0$

if $\varphi$ is supported outside of supp $f$.

the support is the intersection of all closed sets $K \subseteq \mathbb{R}^n$ with the property: $\varphi \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) \subseteq K^c$ $\Rightarrow \langle u, \varphi \rangle = 0$

Example: $\operatorname{supp}(\delta_{x_0}) = \{x_0\}$, $\operatorname{supp}(\partial^\alpha \delta_{x_0}) = \{x_0\}$



$\boxed{\varphi(x_0) = 0}$

$\langle \delta_{x_0}, \varphi \rangle$

## Distribution supported at a point:

**Propn.** If $u \in \mathcal{S}'(\mathbb{R}^n)$ and is supported on $\{x_0\}$, then there is a positive integer $k$, and complex numbers $a_\alpha$ such that,

$$\boxed{u = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_{x_0}}$$

Proof: WLOG, $x_0 = 0$,

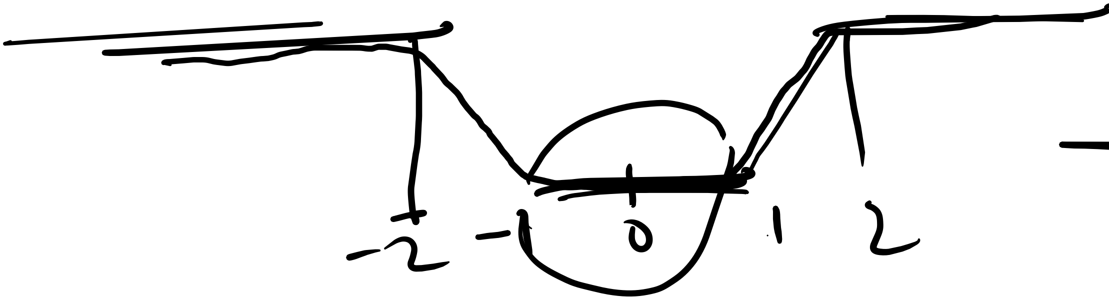
For some $C, \cancel{\mathcal{B}}$, $m, k$

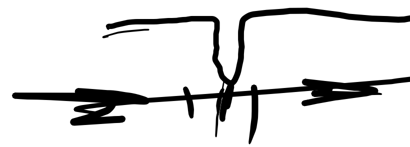
$$\left( |\langle u, f \rangle| \le C \sum_{\substack{|\alpha| \le m \\ |\beta| \le \boxed{k}}} \underbrace{\left( \sup_{x \in \mathbb{R}^n} |x^\alpha (\partial^\beta f)(x)| \right)}_{p_{\alpha,\beta}(f)} \quad \forall f \in \mathcal{S}(\mathbb{R}^n) \right)$$

Step I: claim: $\cancel{\exists f}$ $\varphi \in \mathcal{S}(\mathbb{R}^n)$, satisfying $\boxed{(\partial^\beta \varphi)(0) = 0}$ $\forall |\beta| \le k$

then $\boxed{\langle u, \varphi \rangle = 0.}$

Proof of Claim -.



$f \in C^\infty(\mathbb{R}^n)$, $f(x) = 1$, if $|x| \geq 2$

$f(x) = 0$, if $|x| \leq 1$.

$f^\varepsilon(x) := f\left(\frac{x}{\varepsilon}\right)$, $\boxed{\rho_{\alpha,\beta}}\left(\underline{f^\varepsilon \varphi - \varphi}\right) \to 0$

as $\varepsilon \to 0$

for $|\alpha| \leq m$, $|\beta| \leq k$

$\varphi \in \mathcal{S}(\mathbb{R}^n)$, s.t. $\boxed{\partial^\beta \varphi(0) = 0}$, $\forall |\beta| \leq k$. we have used the hypothesis (and ~~supp~~ supp $u \subset \{0\}$).

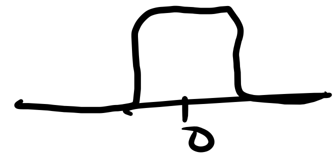
$$|\langle u, \boxed{\varphi} \rangle| \leq \boxed{|\langle u, \rho^\varepsilon \varphi \rangle|} + |\langle u, \rho^\varepsilon \varphi - \varphi \rangle|$$

$$\leq 0 + \left| \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq k}} \underbrace{p_{\alpha,\beta}(\rho^\varepsilon \varphi - \varphi)}_{\to 0 \text{ as } \varepsilon \to 0} \right|$$

$$\Rightarrow \langle u, \varphi \rangle = 0$$

Step II. Proof of proposition.

Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\eta \in C_c^{\infty}(\mathbb{R}^n)$



$\eta \equiv 1$ on a neighbourhood of zero.

$f_k$

$$f(x) = \eta(x)\left( \sum_{|\beta| \le k} \frac{\partial^\beta f(0)}{\beta!} x^\beta + h(x) \right) + (1-\eta(x))\, f(x),$$

where $h(x) = O(|x|^{k+1})$ as $|x| \to 0$.

$$\boxed{\partial^\beta h(0) = 0,} \quad \forall\ |\beta| \le k. \qquad \lim_{x\to 0} \frac{h(x)}{x} = 0$$

By Step I,

$$\langle u, \underline{\eta h} \rangle = 0.$$

$$\partial^\beta(\eta h)(0) = 0, \quad \forall |\beta| \le k.$$

As supp $u \subset \{0\}$, $\quad \langle u, (1-\eta) f \rangle = 0.$

$$\langle u, f \rangle = \sum_{|\beta| \le k} \frac{\partial^\beta f}{\beta!}(0) \cdot \boxed{\langle u, x^\beta \eta(x) \rangle}$$

no dependence on $f$.

$$= \left\langle \sum_{|\beta| \le k} \frac{\langle u, x^\beta \chi(x) \rangle}{\beta!} \partial^\beta \delta_0, \; f \right\rangle$$

$$\boxed{u = \sum_{|\beta| \le k} \boxed{a_\beta} \partial^\beta \delta_0}$$

, where

$$a_\beta = \frac{\langle u, x^\beta \chi(x) \rangle}{\beta!}$$

<u>Corollary</u>: Let $u \in \mathcal{S}'(\mathbb{R}^n)$; if $\operatorname{supp}(\hat{u}) \ll \{\xi_0\}$, then $u$ is a finite linear combination of functions $(-2\pi i x)^{\alpha}$ $\boxed{e^{2\pi i \langle x, \xi_0 \rangle}}$.

In particular, if $\mathrm{supp}(\hat{u}) = \{0\}$, then $u$ is a polynomial function.

<u>The Laplacian</u>: (partial differential operator acting on $\mathcal{S}'(\mathbb{R}^n)$)

$$\Delta(u) := \sum_{j=1}^{n} \partial_j^2 u.$$

Solutions of Laplace's eqn $\boxed{\Delta u = 0}$ are called <u>harmonic distributions</u>.

<u>Corollary</u>. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\Delta u = 0$, then $u$ is a polynomial.

Pf.

$$\Delta u = 0$$

$$\underline{\underline{-4\pi^2 |\xi|^2}}\ \hat{u}(\xi) = 0, \quad \forall \xi \in \mathbb{R}^n.$$

$\Rightarrow$ $\hat{u}$ is supported on $0$

$\Rightarrow$ $u$ is a polynomial.

Corollary 3.1 (Liouville theorem)

Every bounded harmonic function on $\mathbb{R}^n$ must be constant.

Proof 1. Every bounded polynomial function is constant.

$\Delta u = \delta_0$ on $\mathbb{R}^n$

Busemann-Petty problem

origin-symmetric convex body in $\mathbb{R}^n$

$S^{n-1}$