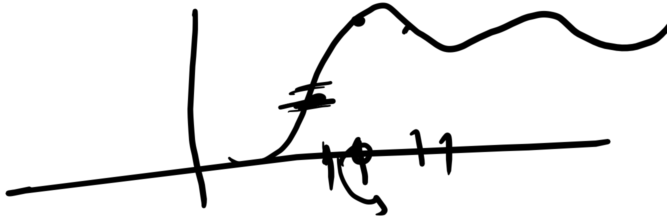
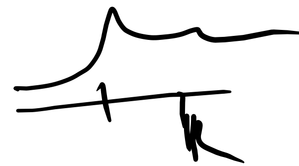
17/08/2021

Lecture - 10

The Poisson Summation Formula:





Let f be an (appropriate) function on $\mathbb{R}$. The function, $\left(\sum_{m\in\mathbb{Z}} f(x+m)\right) = g$, is said to be the periodization of f. (g is a periodic function with period 1)

The restriction of f to $\mathbb{Z}$ (discrete subgroup of $\mathbb{R}$) is said to be (a) <u>discretization</u> of f. (or <u>uniform sampling</u> of f).

Choosing some uniformly spaced points in $\mathbb{R}$ (including 0) and discarding information about the values of f on the rest.

<u>Theorem</u>: Suppose $f \in L^1(\mathbb{R}^n)$.

Then the series $\left(\sum_{m \in \mathbb{Z}} f(x+m)\right)$ converges in the norm of $L^1(\overset{\mathbb{R}/\mathbb{Z}}{S^1})$. The resulting function in $L^1(S^1)$ has the Fourier expansion.

$$\sum_{m \in \mathbb{Z}} \hat{f}(m)\, e^{2\pi i m x}.$$

(that is, $(\hat{f}(m))_{m \in \mathbb{Z}}$ is given the Fourier coefficients of the L^1-function defined by the series $\sum_{m \in \mathbb{Z}} f(x+m)$.)

Proof.

$$\int_0^1 \Big| \sum_{m\in\mathbb{Z}} f(x+m) \Big| \, dx \qquad \sum_{m\in\mathbb{Z}} f(x+m) \in L^1(S^1)$$

$$\leq \int_0^1 \sum_{m\in\mathbb{Z}} | f(x+m) | \, dx$$

$$\leq \sum_{m\in\mathbb{Z}} \int_m^{m+1} |f(x)| \, dx$$

$$\leq \int_{\mathbb{R}} |f(x)| \, dx = \|f\|_1 < \infty$$

Thus, the series $\sum_{m \in \mathbb{Z}} f(x+m)$ converges absolutely in the norm of $\boxed{L^1(S^1)}$

Using DCT we have $g(x)$

$$c_m(g) = \int_0^1 \left(\sum_{m' \in \mathbb{Z}} f(x+m') \right) e^{-2\pi i m x} \, dx$$

$$= \sum_{m' \in \mathbb{Z}} \int_0^1 f(x+m') \, e^{-2\pi i m x} \, dx$$

$$= \sum_{m' \in \mathbb{Z}} \int_{m'}^{m'+1} f(x) \, e^{-2\pi i m (x - m')} \, dx$$

$$= \int_{\mathbb{R}} f(x)\, e^{-2\pi i m x}\, dx$$

$$= \hat{f}(m)$$

$$c_m(\text{periodization of } f) = \boxed{\hat{f}(m)}$$

g

$$\boxed{\mathcal{F} \circ P = U \circ \mathcal{F}}$$

Corollary — (Poisson Summation Formula)

Suppose $\hat{f}(y) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i xy}\, dx$

and $f(x) = \int_{\mathbb{R}} \hat{f}(y)\, e^{2\pi i xy}\, dy$, with

$\boxed{|f(x)| \le \dfrac{A}{(1+|x|)^{1+\delta}}}$, $|\hat{f}(y)| \le \dfrac{A}{(1+|y|)^{1+\delta}}$, $\delta > 0$

(so, that both f and $\hat{f}$ may be assumed to be continuous). Then,

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e^{2\pi i mx}$$

and in particular,

$$\boxed{\sum_{m\in\mathbb{Z}} \boxed{f(m)} = \sum_{m\in\mathbb{Z}} \boxed{\hat{f}(m)}}$$

(And the four series converge absolutely).

Proof: ($h \in L^1(S^1)$, $\boxed{\sum_{k\in\mathbb{Z}} |c_k(h)| < \infty}$ then

$h \in C(S^1)$ and equals $\underline{\sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}}$

Quick proof: $S_K(h) \to \boxed{g}$ uniformly

$\Rightarrow \sigma_K(h) \to g$. pointwise

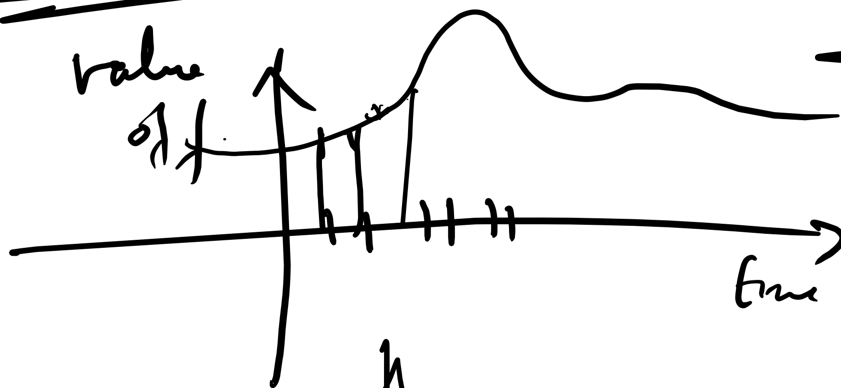
So, $\displaystyle\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{m\in\mathbb{Z}} \hat{f}(m) e^{2\pi i x m}$, almost $\forall x \in [0,1)$.

~~everywhere~~ ~~on [0,1)~~.

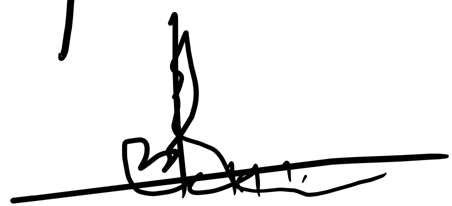
Remark $\mathbb{Z} \hookrightarrow \mathbb{R}$, $\mathbb{R}/\mathbb{Z} \to S^1$

Application (Shannon–Nyquist sampling theorem).

Theorem. If a function $f(t)$ contains no frequencies higher than B hertz (in other words, $\hat{f}$ is supported on $[-B,B]$), then f is completely determined by giving its values at a series of points spaced $\left(\frac{1}{2B}\right)$ seconds apart.



(band-limit)



* A sufficient sample rate is therefore anything larger than 2B samples per second.

$$\left[\sum_{m\in\mathbb{Z}} \hat{f}(x+m) = \sum_{m\in\mathbb{Z}} f(m)\, e^{-2\pi i m x}\right]$$

$[-\frac{1}{2}, \frac{1}{2}]$,

For $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\boxed{\hat{f}(x) = \sum_{m\in\mathbb{Z}} f(m)\, e^{-2\pi i m x}}$$

$$\hat{f}(x) = \sum_{m \in \mathbb{Z}} f(m)\, \mathcal{F}\left(\operatorname{sinc}(\pi(\xi - m))\right)$$

$$h(x) = \boxed{e^{2\pi i m x}}\, \boxed{\chi_{[-\frac{1}{2}, \frac{1}{2}]}}$$

$$\operatorname{sinc} \alpha = \frac{\sin \alpha}{\alpha}$$

$$\hat{h}(\xi) = \int_{-1/2}^{1/2} e^{2\pi i m x} e^{-2\pi i x \xi}\, dx$$

$$= \int_{-1/2}^{1/2} e^{-2\pi i x(\xi - m)}\, dx$$

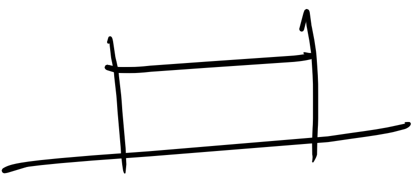
$$= \left. \frac{e^{2\pi i x(\xi - m)}}{-2\pi i(\xi - m)} \right]_{-1/2}^{1/2}$$

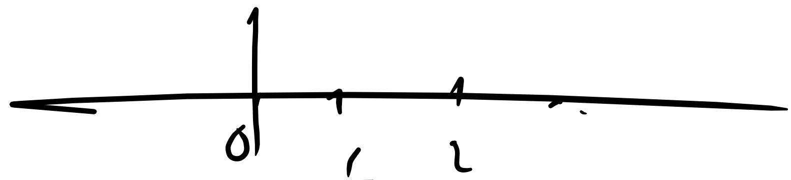
$$\hat{f}(x) = \sum_{m \in \mathbb{Z}} f(m)\, e^{2\pi i m x}\, \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$$

$$\Rightarrow \boxed{f(x) = \sum_{m \in \mathbb{Z}} f(m)\, \mathrm{sinc}\, \pi(x-m)}$$

(Shannon-Whittaker formula)

$$\boxed{f = f * \delta}$$





$\boxed{\mathcal{F}(P)\ \ U \circ f}$

$$\boxed{\mathcal{F}(D^\alpha u) = (2\pi i|x|)^\alpha \cdot \hat{u}.}$$

$$\int_{\mathbb{R}^n} D^\alpha u \, e^{-2\pi i x\xi} \, dx = (-1)^{|\alpha|} (-2\pi i|x|)^\alpha$$

$$(D^2 + I)u = 0$$

$$\frac{d^2u}{dx^2} + u = 0 \Rightarrow (2\pi i x)^2 \hat{u} + \hat{u} = 0$$

$$\Rightarrow \hat{u}(y^2 + 1) = 0 ;$$

$$\Rightarrow \cancel{y = \pm i}$$

$$\boxed{\frac{d^2 u}{dx^2} - 3\frac{du}{dx} + 2u = 0} \quad \boxed{u = e^{\lambda x}}$$

$$\Rightarrow \boxed{(x^2 - 3x + 2)}\,\hat{u}(x) = 0 \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow \text{ if } x \notin \{1, 2\}, \text{ then } \hat{u}(x) = 0,$$

$\Rightarrow \hat{u}$ is "supported" on $\{1, 2\}$.

$$\hat{u} = a_1 \delta_1 + a_2 \delta_2$$

(Fourier transform of $\delta_{a}^{\wedge}(\xi) = e^{-2\pi i a \cdot \xi}$

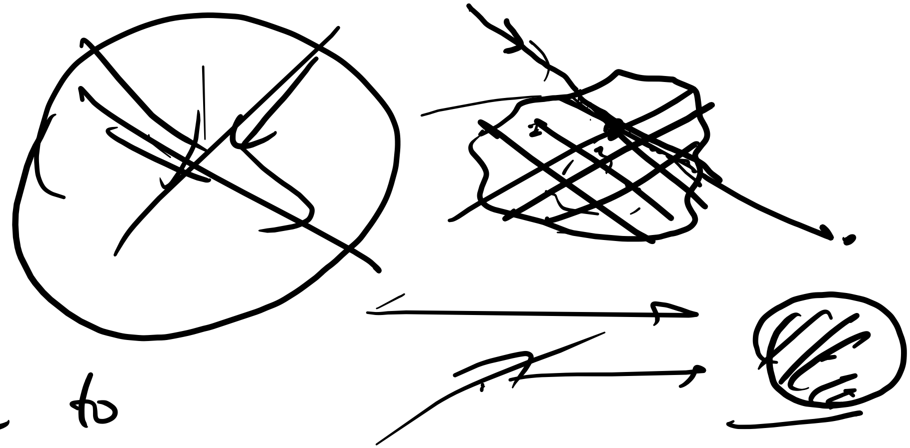
$$a = a_1 e^{\boxed{2\pi i}\xi} + a_2 e^{\boxed{2\pi i}(2\xi)}$$

Theory of Distribution :

Test functions, : (Four blind men and an elephant)

Probe an object from all possible angles and figure out all information about the object.

CAT sense:



For $f \in L^1_{loc}(\mathbb{R}^d)$,

it is not clear how to

interprete value at a point.

$$T_f(\varphi) = \int_{\mathbb{R}} f(x)\,\boxed{\varphi(x)}\,dx$$

weighted average of f

(testing against smooth functions.)

(Inverse problem) $\longrightarrow$ Can we find f from the knowledge of $T_f(-)$

That depends on how much probing has been done, or how large is the space of test functions.

Riesz representation theorem : μ $\quad C(X) \to$ compact Hausdorff space

$$\int f \, d\mu \quad \forall f \in C(X)$$

$\to$ test function.

Spaces of Test Functions:

$$C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n)$$

$C_c^\infty(\mathbb{R}^n)$ → smooth functions with compact support

$\mathcal{S}(\mathbb{R}^n)$ → Schwartz functions

$C^\infty(\mathbb{R}^n)$ → smooth functions

Defn (1) $f_k \to f$ in $C^\infty(\mathbb{R}^n)$

$\iff$ $f_k, f \in C^\infty(\mathbb{R}^n)$ and

$$\lim_{k\to\infty}\Big(\sup_{|x|\leq N}|\partial^\alpha(f_k-f)(x)|\Big) = 0$$

$\forall$ multi-indices α, $N = 1, 2, \ldots$

$\boxed{\rho_{\alpha, N}}$

(2) $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$ $\to \rho_{\alpha,\beta}(f_k - f)$

$$\Leftrightarrow \lim_{k\to\infty}\left(\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta (f_k - f)(x)| \right) = 0$$

$\forall$ α, β multi-indices.

(3) $f_k \to f$ in $C_c^\infty(\mathbb{R}^n)$.

$\Leftrightarrow$ $f_k, f \in C_c^\infty(\mathbb{R}^n)$, $\exists B$ compact in $\mathbb{R}^n$, such that $\boxed{\mathrm{supp}(f_k)} \subseteq B$, $\forall k \in \mathbb{N}$,

and $\lim_{k\to\infty} \|\partial^\alpha (f_k - f)\|_\infty \to 0$ $\forall$ multi-index α.

Spaces of Functionals on Test Functions

$$(C_c^\infty(\mathbb{R}^n))' = \mathcal{D}'(\mathbb{R}^n)$$ (distributions)

$$(\mathcal{S}(\mathbb{R}^n))' = \mathcal{S}'(\mathbb{R}^n)$$ (tempered distributions)

$$(C^\infty(\mathbb{R}^n))' = \mathcal{E}'(\mathbb{R}^n)$$ (distributions with compact support)

$$\mathcal{E}' \subseteq \mathcal{S}' \subseteq \boxed{\mathcal{D}'}$$ "generalized functions"

$T_n \to T$ in $\mathcal{D}'$

$\Leftrightarrow$ T_n, $T \in \mathcal{D}'$ and $T_j(f) \to T(f)$
$\forall f \in C_c^\infty(\mathbb{R}^n)$

(weak-* topology on $\mathcal{D}'$)

Notation: $\langle u, f \rangle$ is $u(f)$

u: distribution or tempered distribution

f: test function.

Prop^n: (a) A linear functional u on $C_c^\infty(\mathbb{R}^n)$ is a distribution iff for every compact $K \subseteq \mathbb{R}^n$, $\exists$ $C > 0$ and $m \in \mathbb{Z}$ s.t.

$$|\langle u, f \rangle| \leq C \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_\infty \right)$$

$\forall f \in C_c^\infty(\mathbb{R}^n)$ with support in K.

(b)

(b) A linear functional u on $\mathcal{S}(\mathbb{R}^n)$ is a <u>tempered distribution</u> if and only if

(1) $\exists\, C > 0$ and $k, m \in \mathbb{N}$ s.t.

$$|\langle u, f\rangle| \le C \left(\sum_{\substack{|\alpha| \le m \\ |\beta| \le k}} \rho_{\alpha,\beta}(f) \right)$$

$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$

Proof (b) $\{ f \in \mathcal{S}(\mathbb{R}^n) : \rho_{\alpha,\beta}(f) < \delta \}$ forms a subbase for the topology.

$\exists \delta > 0$, s.t.

$$p_{\alpha,\beta}(f) < \delta \Rightarrow |\langle u, f \rangle| \leq 1.$$

$\Leftarrow$ $\quad p_{\alpha,\beta}(f) \leq \frac{\delta}{2} \Rightarrow |\langle u, f \rangle| \leq 1$

Choose, $\quad C = \frac{2}{\delta}$.

$$\Rightarrow p_{\alpha,\beta}\left(\frac{2}{\delta} f\right) \leq 1 \Rightarrow \left|\langle u, \frac{2}{\delta} f \rangle\right| \leq \boxed{\frac{2}{\delta}}$$

Examples -

(1) Dirac mass at origin (δ_0).

$$\langle \delta_0, \varphi \rangle := \varphi(0)$$

$$\langle \delta_a, \underline{\varphi} \rangle := \varphi(a), \quad [a \in \mathbb{R}^n]$$

Verify δ is a "distribution with compact support".

(2) If $f \in \boxed{L^1_{loc}(\mathbb{R}^n)}$, then $f \in \mathcal{D}'(\mathbb{R}^n)$.

$$\boxed{L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx}$$

distribution corresponding to f.

If $L_{f_1} = L_{f_2} \Rightarrow L_f = 0$, where $f = f_1 - f_2$

(3) $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, are contained in $S'(\mathbb{R}^n)$.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$$

$\mathbb{R}^2$, $\Delta u = f$

$\mathbb{R}^3$ $\quad \dfrac{1}{(x_1^2+x_2^2+x_3^2)^{1/2}}$

$$\left((2\pi i \xi_1)^2 + (2\pi i \xi_2)^2\right)(\hat{u}) = 1$$

$$\hat{u}(\xi_1, \xi_2)(\xi_1^2+\xi_2^2) = 0$$

$\dfrac{1}{r^2}$ $\quad \hat{u} = \dfrac{1}{\xi_1^2+\xi_2^2}$

$\hat{u}$ is supported on $\{0\}$

$u = \log|x|$

$= \log\sqrt{x_1^2+x_2^2}$

$\hat{u} =$