Analysis 2 - JRF

Assignment 6 — Even Semester 2020-2021

Due date: September 09, 2021 (by 11:59 pm)
Note: Each question is worth 10 points.

We denote the space of Schwartz functions by $\mathscr{S}(\mathbb{R}^n)$, and the space of tempered distributions on $\mathbb{R}^n$ by $\mathscr{S}'(\mathbb{R}^n)$. The dilation of a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ by $a > 0$ is denoted by $\delta_a(u)$.

The mean value of a function u taken over the surface of a sphere of radius $r > 0$ about the point $x \in \mathbb{R}^n$ is denoted by,

$$\mathscr{M}_{x,u}(r) := \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x + rt')\, dt'.$$

We denote the open ball of radius r in $\mathbb{R}^n$ by $B_n(0, r)$.

1. Let Ω be a domain in $\mathbb{R}^n$. We say that $u \in C^2(\overline{\Omega})$ is **subharmonic** if $\Delta u \geq 0$.
   (a) Show that for every $x \in \Omega$ and $r > 0$ such that $B(x, r) \subset \Omega$, we have
   $$u(x) \leq \mathscr{M}_{x,u}(r).$$
   (b) Use part (a) to prove the maximum principle for subharmonic functions:
   $$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

2. Let Ω be a domain in $\mathbb{R}^n$ and u be harmonic in Ω.
   (a) For a convex function $\varphi \in C^\infty(\mathbb{R})$ and u be harmonic in Ω. Prove that $\varphi \circ u$ is subharmonic.
   (b) Let u be harmonic in Ω. Prove that $|\nabla u|^2$ is subharmonic.

3. Let u be a non-negative function which is harmonic in $B_n(0, 1)$ with a continuous extension to its boundary.
   (a) Show that for every x such that $|x| < 1$, we have
   $$\frac{1 - |x|}{(1 + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{1 + |x|}{(1 - |x|)^{n-1}} u(0).$$
   (b) For $0 \leq r \leq \frac{1}{2}$, show that
   $$\sup_{B(0,r)} u \leq 3^n \inf_{B(0,r)} u.$$

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4. Let u_k be an increasing sequence of functions which are harmonic in $B_n(0,1)$. Assume that $u_n(0)$ is a Cauchy sequence. Show that there exists a harmonic function u such that $u_n \to u$ uniformly in $B_n(0,r)$ for every $r < 1$.

5. A distribution in $\mathscr{S}'(\mathbb{R}^n)$ is called *homogeneous of degree* $\gamma \in \mathbb{C}$ if

$$\langle u, \delta_\lambda(f)\rangle = \lambda^{-n-\gamma}\langle u, f\rangle, \forall \lambda > 0.$$

   (a) Prove that this definition agrees with the usual definition of homogeneity for functions.

   (b) Show that δ_0 is homogenous of degree $-n$.

   (c) Show that u is homogeneous of degree γ if and only if $\widehat{u}$ is homogeneous of degree $-n-\gamma$.

   Remark: In particular, the Fourier transform of a homogeneous distribution of degree $-n+1$ is a homogeneous distribution of degree -1. For an origin-symmetric convex body K, let $\|x\|_K$ denotes the Minkowski functional of K. In the context of our discussion on the Buseman-Petty problem, recall that the formula for central cross-sectional area is based on the Fourier transform of $\|x\|_K^{-n+1}$, which is a homogeneous distribution of degree $-n+1$.

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