Analysis 2 - JRF

Assignment 5 — Even Semester 2020-2021

Due date: August 30, 2021 (by 11:59 pm)
Note: Each question is worth 10 points.

We denote the space of Schwartz functions by $\mathscr{S}(\mathbb{R}^n)$, and the space of tempered distributions on $\mathbb{R}^n$ by $\mathscr{S}'(\mathbb{R}^n)$. The translation of a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ by a is denoted by $\tau_a u$.

1. Suppose that a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies the estimate:
$$|f(x)| \leq \frac{C}{(1+|x|)^N},$$
for some $N > n$. Show that $\hat{f} \in C^m(\mathbb{R}^n)$ for $1 \leq m \leq [N-n]$.

2. For multi-indices α, β, note that the tempered distribution $u := x^\alpha \partial^\beta \delta_0$ is supported at the origin. Express u as a complex linear combination of derivatives of δ_0.

3. (a) Prove that e^x is not in $\mathscr{S}'(\mathbb{R})$ but that $e^x e^{ie^x}$ is in $\mathscr{S}'(\mathbb{R})$.
   (b) Compute the derivative of the tempered distribution $e^x e^{ie^x}$.

4. For $\varepsilon > 0$, let $T_\varepsilon := \frac{\varepsilon}{2}|x|^{\varepsilon-1}$. Compute in $\mathscr{S}'(\mathbb{R})$, the following limit:
$$\lim_{\varepsilon \to 0} T_\varepsilon.$$

5. (a) Let p be a positive integer and $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that
$$|a_n| \leq Cn^p, \forall n \in \mathbb{N}.$$
Prove that the sequence $T_N := \sum_{n=-N}^{N} a_n e^{2\pi i n x}$ converges in $\mathscr{S}'(\mathbb{R}^n)$.
   (b) Let $T := \lim_{N\to\infty} T_N$. Show that
$$\frac{dT}{dx} = \sum_{-\infty}^{\infty} (2\pi i n) a_n e^{2\pi i n x},$$
and that $\tau_1 T = T$.

6. (Poisson Summation formula) Show that $\sum_{-\infty}^{\infty} e^{2\pi i n x} = \sum_{-\infty}^{\infty} \delta_n$ in $\mathscr{S}'(\mathbb{R})$.

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