Analysis 2 - JRF

Assignment 4 — Even Semester 2020-2021

Due date: August 23, 2021 (by 11:59 pm)

Note: Each question is worth 10 points.

1. (a) Prove that for all $0 < \varepsilon < t < \infty$ we have

$$\Big|\int_{\varepsilon}^{t} \frac{\sin(\xi)}{\xi}\, d\xi\Big| \leq 4$$

.

   (b) If f is an odd L^1 function on $\mathbb{R}$, show that for all $t > \varepsilon > 0$ we have

$$\Big|\int_{\varepsilon}^{t} \frac{\hat{f}(\xi)}{\xi}\, d\xi\Big| \leq 4\|f\|_{L^1}.$$

   (c) Let $g(\xi)$ be a continuous odd function that is equal to $\frac{1}{\log \xi}$ for $\xi \geq 2$. Show that there does not exist an L^1-function whose Fourier transform is g.

2. (a) Let f be in $L^1(\mathbb{R})$. Prove that

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f\Big(x - \frac{1}{x}\Big)\, dx.$$

   (b) Using part (a) for $f(x) = e^{-tx^2}$, prove the identity:

$$e^{-2t} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y-\frac{t^2}{y}}\, \frac{dy}{\sqrt{y}}, \text{ where } t > 0.$$

3. (Calculation of the Fourier transform of the Poisson kernel) Set $t = \pi|x|$ in exercise 2(b) and integrate with respect to $e^{-2\pi i \xi \cdot x}\, dx$ to prove that

$$(e^{-2\pi|x|})\widehat{\ }(\xi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

4. Show that if $g \in C_c^{\infty}(\mathbb{R})$ and $f \in L^1_{\text{loc}}(\mathbb{R})$, then $f * g$ exists and is infinitely differentiable on $\mathbb{R}$.

5. Let X be a measure space.

   (a) For $0 < p_0 < p_1 \leq \infty$, let $f \in L^{p_0}(X) \cap L^{p_1}(X)$. Then for all $p_0 \leq p \leq p_1$, show that $f \in L^p(X)$.

   (b) For $0 \leq \theta \leq 1$, and $\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, show that

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}.$$

Soumyashant Nayak: soumyashant@isibang.ac.in