Analysis 2 - JRF

Assignment 3 — Even Semester 2020-2021

Due date: August 16, 2021 (by 11:59 pm)
Note: Each question is worth 10 points.

For $n \in \mathbb{N}$, we denote the space of Schwartz functions on $\mathbb{R}^n$ by $\mathscr{S}(\mathbb{R}^n)$, and the space of smooth functions on $\mathbb{R}^n$ with compact support by $C_c^\infty(\mathbb{R}^n)$. For a measurable function f on $(\mathbb{R}^n; \mu)$ (where μ is the Lebesgue measure), we denote the centred Hardy-Littlewood maximal function by $M_c(f)$.

1. If $f \in \mathscr{S}(\mathbb{R}^n)$ and $f(0) = 0$, show that there are functions $g_1, g_2, \ldots, g_n \in \mathscr{S}(\mathbb{R}^n)$ such that
$$f(x) = \sum_{k=1}^{n} x_k g_k(x).$$

2. Let $(a_j)_{j\in\mathbb{N}}$ be a given sequence of complex numbers.
   (a) Let $\varphi \in C_c^\infty(\mathbb{R})$ be supported in $(-2, 2)$ and identically equal to 1 on $[-1, 1]$. Prove that we can find a sequence of real numbers, $(\lambda_k)_{k\in\mathbb{N}}$, such that if we set
$$f_k(x) := \frac{a_k}{k!} x^k \varphi(\lambda_k x),$$
   then
$$\sup_{x\in\mathbb{R}} \left| \left(\frac{d}{dx}\right)^m f_k(x) \right| \leq \frac{1}{2^k}, \text{ for } 1 \leq m \leq k-1.$$
   (b) Prove that the series $\sum_{k=0}^{\infty} f_k(x)$ defines a function f which is smooth and
$$\left(\frac{d^j}{dx^j} f\right)(0) = a_j, j = 0, 1, 2, \ldots$$

3. (Sharpness of the Hardy-Littlewood maximal inequality) Let $f \in L^1(\mathbb{R}^n; \mu)$ be non-zero. Show that there is a constant $\alpha_f > 0$ (depending on f) such that
$$M_c(f)(x) \geq \frac{\alpha_f}{|x|^n},$$
   for sufficiently large values of $|x|$. Conclude that $M_c(f) \notin L^1(\mathbb{R}^n; \mu)$.

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4. (Local non-integrability of $M_c(f)$)

   (a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

   $$f(x) = \begin{cases} \frac{1}{|x| \log(1/|x|)^2} & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that $f \in L^1(\mathbb{R}^n; \mu)$.

   (b) Show that there is a constant $\alpha > 0$ such that

   $$M_c(f)(x) \geq \frac{\alpha}{|x| \log(1/|x|)},$$

   for all $|x| \leq \frac{1}{2}$. Conclude that f is **not** integrable on any neighbourhood of the origin.

5. (Lebesgue density) The *Lebesgue density* $\mathcal{D}_E$ of a measurable subset $E \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined as

   $$\mathcal{D}_E(x) = \lim_{\mu(B) \to 0, x \in B} \frac{\mu(E \cap B)}{\mu(B)},$$

   whenever the limit exists. The set of points $x \in \mathbb{R}^n$ for which $\mathcal{D}_E(x) = 1$ is called the *set of density points* of E.

   (a) Show that $\mathcal{D}_E(x) = 1$ for almost every $x \in E$, and $\mathcal{D}_E(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus E$.

   (b) If $E \subseteq [0, 1]$ is Lebesgue-measurable and there exists $\alpha > 0$ such that $\mu(E \cap I) \geq \alpha \mu(I)$ for all subintervals I of $[0, 1]$, then $\mu(E) = 1$.

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