Analysis 2 - JRF

Assignment 2 — Even Semester 2020-2021

Due date: August 7, 2021 (by 11:59 pm)
Note: Each question is worth 10 points.

Let μ denote the Lebesgue measure on S^1 normalized to $\mu(S^1) = 1$. For $f \in L^1(S^1, \mu)$, let

$$c_n(f) := \int_{S^1} f(z) z^{-n} \, d\mu(z),$$

denote the nth Fourier coefficient of f.

1. Let H be a Hilbert space. Prove the following polarization identity for every $x, y \in H$:

$$4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

2. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$Tf(n) = \begin{cases} f(n-1) & \text{if } n > 1 \\ 0 & \text{if } n = 1. \end{cases}$$

   Show that T is an isometry but not a unitary.

3. For which $s \in \mathbb{C}$ does the function $f(n) = n^{-s}$ belong to $\ell^2(\mathbb{N})$? For which f does it belong to $\ell^1(\mathbb{N})$?

4. Show that if $f \in L^2(S^1, \mu)$, then

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|c_k(f)|}{|k|} < \infty.$$

5. Compute the Fourier series of the periodic function f (of period 1) given by: $f(x) = (x - \frac{1}{2})^2$. Use Parseval's formula to prove that:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

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