Analysis 2 - JRF

Assignment 1 — Even Semester 2020-2021

Due date: July 28, 2021 (by 11:59 pm)
Note: Each question is worth 10 points.

Let μ denote the Lebesgue measure on S^1 normalized to $\mu(S^1) = 1$. For $f \in L^1(S^1, \mu)$, let

$$c_n(f) := \int_{S^1} f(z) z^{-n} \, d\mu(z),$$

denote the nth Fourier coefficient of f. We will identify functions on S^1 with periodic functions on $\mathbb{R}$ of period 1 in the natural manner (and use these descriptions interchangeably whenever convenient). The nth partial sum of the Fourier series of f is denoted by

$$S_n(f)(z) = \sum_{k=-n}^{n} c_k(f) z^k, \text{ for } z \in S^1.$$

The set of functions on $\mathbb{Z}$ which vanish at infinity, is denoted by $\ell_0(\mathbb{Z})$. We define $e_k(x) := \exp(2\pi i k x)$ for $x \in [0, 1]$ (or by the identification mentioned above, $e_k(z) = z^n$ for $z \in S^1$).

1. For $f \in L^1(S^1, \mu)$, show that,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{c_k(f) - c_{-k}(f)}{k},$$

exists. Conclude that the element $(a_k)_{k \in \mathbb{Z}}$ of $\ell_0(\mathbb{Z})$ given by

$$a_k = \begin{cases} 0 & \text{if } k \leq 1 \\ \frac{1}{\ln k} & \text{if } k \geq 2 \end{cases}$$

is not the Fourier transform of any function in $L^1(S^1, \mu)$. (Thus, the image of the Fourier transform $\hat{\cdot} : L^1(S^1, \mu) \to \ell_0(\mathbb{Z})$ is a proper subset of $\ell_0(\mathbb{Z})$.)
(**Hint:** Consider the Fourier coefficients of the continuous periodic function $F(x) = \int_0^x \big(f(t) - c_0(f)\big) \, d\mu$.)

2. For $n \in \mathbb{N}$, let $D_n := \sum_{k=-n}^{n} e_k$ denote the Dirichlet kernel. Show that $\|D_n\|_1 \geq C_1 \ln n$ for a fixed constant$C_1 > 0$.

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3. For $f \in C^1(S^1)$, show that

$$\|f - S_n(f)\|_\infty = o\Big(\frac{1}{\sqrt{n}}\Big).$$

4. (a) Give an example of a sequence (f_n) in $L^1(S^1, \mu)$ that converges in the L^1-norm but does not converge pointwise.

   (b) Give an example of a sequence (f_n) in $L^1(S^1, \mu)$ that converges pointwise but does not converge in the L^1-norm.

5. Compute the Fourier series of the periodic function f (of period 1) given by:

   (a) $f(x) = |x|$ for $-\frac{1}{2} \le x \le \frac{1}{2}$;

   (b) $f(x) = x$ for $x \in [0, 1)$;

   (c) $f(x) = e^{ax}$ for $x \in [0, 1)$.

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