Analysis I - JRF

Assignment 1 — 1st Semester 2022-2023

Due date: October 07, 2022 (in Moodle)
Note: Total number of points is 70. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

Let X be a measure space.

1. (10 points) (a) (5 points) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x \in X : f(x) < g(x)\}, \{x \in X : f(x) = g(x)\},$$

   are measurable.

   (b) (5 points) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

2. (10 points) Suppose μ is a positive measure on X, $f : X \to [0, \infty)$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and $\alpha > 0$ is a constant. Evaluate the following limit (with justification):

$$\lim_{n\to\infty} \int_X n \log\left(1 + (f/n)^\alpha\right) d\mu.$$

3. (10 points) Suppose $f \in L^1(X; \mu)$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

4. (10 points) Suppose $\mu(X) < \infty$, and let $\{f_n\}$ be a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

   and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

5. (10 points) For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n := \int_{[n,n+1)} f \, dx$ for each $n \in \mathbb{N}$: Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges?

Soumyashant Nayak: soumyashant@isibang.ac.in

6. (10 points) Let $-\infty < a < b < \infty$ and let $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}$ be such that $f(\vec{x},t)$ is integrable for any $t \in [a,b]$. Let

$$F(t) = \int_{\mathbb{R}^n} f(\vec{x},t)\, d\vec{x}.$$

   (a) (5 points) Suppose that $f(\vec{x},\cdot)$ is continuous in t for every $\vec{x} \in \mathbb{R}^n$ and that there exists an integrable function g such that $|f(\vec{x},t)| \leq g(\vec{x})$ for all $\vec{x}, t$. Show that the function $F(t)$ is continuous.

   (b) (5 points) Suppose that $\frac{\partial f}{\partial t}$ exists and that there exists an integrable function h such that $|\frac{\partial f}{\partial t}(\vec{x},t)| \leq h(\vec{x})$ for all $\vec{x}, t$. Then the function $F(t)$ is differentiable and

$$F'(t) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial t}(\vec{x},t)\, d\vec{x}.$$

7. (10 points) Consider the functions

$$f(t) = \left(\int_{[0,t]} e^{-x^2}\, dx\right)^2, \;\; g(t) = \int_{[0,1]} \frac{e^{-t^2(1+x^2)}}{1+x^2}\, dx.$$

   (i) Show that $f'(t) + g'(t) = 0$ for all $t \geq 0$;

   (ii) Show that $f(t) + g(t) = \frac{\pi}{4}$ for all $t \geq 0$;

   (iii) Conclude that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

Soumyashant Nayak: soumyashant@isibang.ac.in