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CLASS - X

Schur - Horn Theorem $\vec{v}, \vec{w} \in \mathbb{R}^n$. Then $\vec{v}$ is the diagonal of a Hermitian matrix A with eigenvalues (counting multiplicity) given by $\vec{w}$ iff $\vec{v} \prec \vec{w}$.

Hadamard Determinant Inequality

Hadamard - Fischer Determinant Inequality

Convex Monotone Functions

$f: \mathbb{R} \to \mathbb{R}$ be any function. $f(\vec{x}) \equiv (f(x_1), \ldots, f(x_n))$, $\vec{x} \in \mathbb{R}^n$.

Theorem: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then the followings are equivalent.

(i) $\vec{x} \prec \vec{y}$

(ii) $tr(\phi(\vec{x})) \le tr(\phi(\vec{y}))$ $\forall$ convex function $\phi: \mathbb{R} \to \mathbb{R}$. $[tr(\vec{x}) = x_1 + \ldots + x_n]$

(i.e. $\phi(x_1) + \ldots + \phi(x_n) \le \phi(y_1) + \ldots + \phi(y_n)$)

Proof:- $\boxed{(ii) \Rightarrow (i)}$ is immediate using $\phi_t(x) = |x - t|$, $t \in \mathbb{R}$.

$\boxed{(i) \Rightarrow (ii)}$ $\vec{x} = A\vec{y}$ for some Doubly stochastic matrix A.

$x_i = \sum_{j=1}^{n} a_{ij} y_j \quad a_{ij} \ge 0$.

$\phi(x_i) \le \sum_{j=1}^{n} a_{ij} \phi(y_j)$ [By Jensen's Inequality].

$\sum_{i=1}^{n} \phi(x_i) \le \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}\right) \phi(y_j) \Rightarrow \sum_{i=1}^{n} \phi(x_i) \le \sum_{j=1}^{n} \phi(y_j)$

Example.

$\phi \equiv -\log$ (Convex function)

$A \to$ Hermitian matrix (Positive Definite)

$(a_{11}, \ldots, a_{nn}) \prec (\lambda_1, \ldots, \lambda_n)$

$\phi(a_{11}) + \cdots + \phi(a_{nn}) \leq \phi(\lambda_1) + \cdots + \phi(\lambda_{nn})$.

$\Rightarrow -\log(a_{11} \cdots a_{nn}) \leq -\log(\lambda_1 \cdots \lambda_n)$

$\Rightarrow \lambda_1 \cdots \lambda_n \leq a_{11} \cdots a_{nn}$.

$\Rightarrow \det A \leq a_{11} \cdots a_{nn}$.

$A = (a_{ij})_{1 \leq i,j \leq n}$. $A^* A = \begin{bmatrix} \sum_{j=1}^{n} |a_{j1}|^2 & & & \\ & \sum_{j=1}^{n} |a_{j2}|^2 & & \\ & & \ddots & \\ & & & \sum_{j=1}^{n} |a_{jn}|^2 \end{bmatrix}$

$|\det A| \leq \prod_{j=1}^{n} \sqrt{\sum_{j=1}^{n} |a_{ji}|^2}$

volume of n-parallelopiped. [equality holds if and only if A^*A is already diagonal assuming A is invertible]

Theorem Let $\phi: \mathbb{R} \to \mathbb{R}$ be a convex function. Then the induced map $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ ($\Phi(x_1, \ldots, x_n) := (\phi(x_1), \ldots, \phi(x_n))$). If $\vec{x} \prec \vec{y}$, $\Phi(\vec{x}) \prec_\omega \Phi(\vec{y})$. If ϕ is also monotonically increasing, then $\vec{x} \prec_\omega \vec{y} \Rightarrow \Phi(\vec{x}) \prec_\omega \Phi(\vec{y})$.

Proof:- $\exists$ permutation matrices $P_1, \ldots, P_N$ & $t_i \geq 0$, $\sum_{i=1}^{N} t_i = 1$ s.t. $\vec{x} = \left(\sum_{j=1}^{N} t_j P_j\right) \vec{y}$

$\Phi(\vec{x}) \leq \sum_{j=1}^{N} t_j \Phi(P_j \vec{y})$ (Coordinate wise) [By Jensen's Inequality]
$= z$.

$z = \sum_{j=1}^{N} t_j P_j \Phi(\vec{y})$.

So, $z \prec \Phi(\vec{y})$ as $\sum_{j=1}^{N} t_j P_j$ is doubly stochastic.

$\Phi(\vec{x}) \leq z \prec \Phi(\vec{y}) \prec \Phi(\vec{y})$ $\left[\begin{array}{c} \vec{x} \leq \vec{y} \\ \Downarrow \\ \vec{x} \prec_\omega \vec{y} \end{array}\right]$

$\Rightarrow \Phi(\vec{x}) \prec_\omega \Phi(\vec{y})$.

$\vec{x} \prec_w \vec{y}$ iff $\vec{x} \prec \vec{z} \leq \vec{y}$ for some $\vec{z} \in \mathbb{R}^n$.

$x_1 \leq y_1$

$\vdots$

$x_1 + \dots + x_n \leq y_1 + \dots + y_n$.

$z_1 = y_1,\ z_2 = y_2, \dots, z_{n-1} = y_{n-1}$
& $z_n = -(y_1 + \dots + y_n) + (x_1 + \dots + x_{n-1}) + y_n$.

$z_n \leq y_n$.

$A\vec{x} \leqslant \vec{y}$

Cor:- (i) $\vec{x} \prec \vec{y}$ in $\mathbb{R}^n \Rightarrow |\vec{x}| \prec_w |\vec{y}|$

(ii) $\vec{x} \prec \vec{y}$ in $\mathbb{R}^n \Rightarrow (|x_1|^2, \dots, |x_n|^2) \prec_w (|y_1|^2, \dots, |y_n|^2)$

(iii) $\vec{x} \prec \vec{y}$ in $\mathbb{R}^n_{\geq 0} \Rightarrow (x_1^p, \dots, x_n^p) \prec_w (y_1^p, \dots, y_n^p)$ $\quad p > 1$

(iv) $\vec{x} \prec_w \vec{y}$ in $\mathbb{R}^n \Rightarrow \vec{x}^+ \prec_w \vec{y}^+$

(v) If ϕ is any function such that $\phi(e^t)$ is convex & monotonically increasing in t, then
$\log \vec{x} \prec_w \log \vec{y}$ in $\mathbb{R}^n_{>0} \Rightarrow \Phi(\vec{x}) \prec_w \Phi(\vec{y})$

~~$\phi(t) = \log t$~~ $\log \vec{x} \prec_w \log \vec{y}$ in $\mathbb{R}^n_{>0} \Rightarrow (x_1^p, \dots, x_n^p) \prec_w (y_1^p, \dots, y_n^p)$
$0 < p < \infty$.

Example. For $\vec{x} \in \mathbb{R}^n_{>0}$, let $H(\vec{x}) = -\sum_{j=1}^{n} x_j \log x_j$; $\operatorname{tr} \phi(\vec{x})$

$t \longmapsto t \log t$ is a convex function. $\quad \phi : t \to -t \log t$. $\;(\phi''(t) = \frac{1}{t} > 0)$

$\vec{x} \prec \vec{y} \Rightarrow -H(\vec{x}) \prec_w -H(\vec{y})$. If $x_j \geq 0$ and $\sum x_j = 1$.

$[H(1, 0, \dots, 0) \leq H(x_1, \dots, x_n) \leq H(\frac{1}{n}, \dots, \frac{1}{n})]$

H is the entropy of the distribution.

$\boxed{\text{Qs.}}$ $H(\vec{x}) \prec_w H(\vec{y}) \overset{??}{\Rightarrow} \vec{x} \prec \vec{y}$

$\phi : M_n(\mathbb{C}) \to D_n(\mathbb{C})$ $\quad \begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$

Properties: (i) $\phi(\phi(A)) = \phi(A)$

(ii) $\phi(BD) = \phi(B)D$ $\quad$ for $B \in M_n(\mathbb{C})$, $D \in D_n(\mathbb{C})$

(iii) $\phi(DB) = D\phi(B)$

ϕ is a D_n D_n bimodule map.

(iii) $\operatorname{tr}(\phi(A)) = \operatorname{tr}(A) \quad \forall A \in M_n(\mathbb{C})$.

A linear map $\phi : M_n(\mathbb{C}) \to D_n(\mathbb{C})$ with properties (i)-(iii) is unique.

ϕ is said to be a trace-preserving conditional expectation from $M_n(\mathbb{C})$ to $D_n(\mathbb{C})$ ($D_n(\mathbb{C})$ is a unital * subalgebra of $M_n(\mathbb{C})$).

$(M_n(\mathbb{C}), \|\cdot\|_F)$ is a Hilbert Space.

$\mathcal{S} \subseteq M_n(\mathbb{C})$ is a closed subspace of $M_n(\mathbb{C})$.

$P_{\mathcal{S}} \to$ orthogonal projection onto $\mathcal{S}$ gives $\phi_{\mathcal{S}}$.

Theorem Let $\phi : M_n(\mathbb{C}) \to \mathcal{S}$ ($\mathcal{S} \to$ unital * subalgebra of $M_n(\mathbb{C})$) be a trace-preserving conditional expectation onto $\mathcal{S}$. Then for every positive definite matrix $A \in M_n(\mathbb{C})$, $\det A \leq \det \phi(A)$. [with equality iff $A = \phi(A)$]

Example- $\mathcal{S} = D_n(\mathbb{C})$. $\det A \leq a_{11} \cdots a_{nn}$.

$$A = \left[\begin{array}{c|c} A_{11} & * \\ \hline * & A_{22} \end{array}\right] \qquad \det(A) \leq \det(A_{11}) \det(A_{22})$$

(Hadamard - Fischer Inequality).

Proof:- A is positive definite $\Rightarrow \phi(A)$ is positive definite.

$\phi(A)^{-1/2} A \, \phi(A)^{-1/2}$ is a positive definite matrix.

$$\det\left(\phi(A)^{-1/2} A \, \phi(A)^{-1/2}\right) \leq \frac{1}{n}\operatorname{tr}\left(\phi(A)^{-1/2} A \, \phi(A)^{-1/2}\right)$$

$$= \frac{1}{n}\operatorname{tr}\left(A \, \phi(A)^{-1}\right)$$

$$= \frac{1}{n}\operatorname{tr}\left(\phi(A \, \phi(A)^{-1})\right)$$

$$= \frac{1}{n}\operatorname{tr}\left(\phi(A) \, \phi(A)^{-1}\right)$$

$$= 1.$$

$\Rightarrow \det(A) \leq \det[\phi(A)]$

$\Rightarrow \det(A) \leq a_{11} \cdots a_{nn}$.

with equality iff $\phi(A)^{-1/2} A \, \phi(A)^{-1/2} = I$

$\Leftrightarrow A = \phi(A)$

$\Leftrightarrow A$ is diagonal

$X \sim N(0, \Sigma)$ $\quad f(X) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}\langle \vec{x}, \Sigma^{-1}\vec{x}\rangle}$

$H(X) = \int_{\mathbb{R}^n} -f \log f \, dx$ entropy of X. $X = (x_1, \dots, x_n)$.

$= c \log(\det \Sigma)$.

$H(x_1, \dots, x_k) = c \log \det \Sigma_{11}$

$H(x_{k+1}, \dots, x_n) = c \log \det \Sigma_{22}$.

$H(x_1, \dots, x_n) \leq H(x_1, \dots, x_k) + H(x_{k+1}, \dots, x_n)$.

with equality iff $(x_1, \dots, x_k)$ is independent of $(x_{k+1}, \dots, x_n)$

$(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

$X: \Omega \to \mathbb{R}^n$ finite expectation Random Variable.

$\mathcal{H} \subseteq \mathcal{F}$ sub sigma algebra

$E(X | \mathcal{H})$ is averaging of X on a level set of $\mathcal{Y}$.

(I) $E(E(X | \mathcal{H})) = E(X)$ (trace - preserving)

(II) $E(XY | \mathcal{H}) = X E(Y | \mathcal{H})$ if X is $\mathcal{H}$ measurable.

$(X \longmapsto E(X | \mathcal{H}))$ is a bimodule map.

(space of $\mathcal{H}$ measurable functions is a unital $*$ subalgebra of $\mathcal{F}$ measurable functions with finite expectation)

(III) $E(E(X | \mathcal{H}) | \mathcal{H}) = E(X | \mathcal{H})$