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CLASS - IX

$\vec{x} \prec \vec{y}$ $\quad \vec{x}, \vec{y} \in \mathbb{R}^n$ $\quad x_1^{\downarrow} \le y_1^{\downarrow}$

$$x_1^{\downarrow} + \dots + x_k^{\downarrow} \le y_1^{\downarrow} + \dots + y_k^{\downarrow}$$

$$\& \quad x_1^{\downarrow} + \dots + x_n^{\downarrow} = y_1^{\downarrow} + \dots + y_n^{\downarrow}.$$

'$\prec$' (pre order (reflexive & transitive) not symmetric)

At the level of $\mathbb{R}^n_{sym}$, '$\prec$' is a partial order.

$x \prec_w y$ (weak submajorization)

$$\sum_{i=1}^{k} x_i^{\downarrow} \le \sum_{i=1}^{k} y_i^{\downarrow} \quad 1 \le k \le n.$$

$x \prec^w y$ (super majorization (weak))

$$\sum_{i=1}^{k} x_i^{\uparrow} \ge \sum_{i=1}^{k} y_i^{\uparrow} \quad 1 \le k \le n.$$

$\vec{x}^+ = (\max(x_1, 0), \dots, \max(x_n, 0))$

$|\vec{x}| = (|x_1|, \dots, |x_n|)$ $\qquad \vec{x}^+ + (-\vec{x})^+ = |\vec{x}|$

Theorem: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

(i) $\vec{x} \prec_w \vec{y}$ iff $\forall t \in \mathbb{R}$, $\sum_{j=1}^{n} (x_j - t)^+ \le \sum_{j=1}^{n} (y_j - t)^+$

(ii) Let $\vec{x} \prec^w \vec{y}$ iff $\forall t \in \mathbb{R}$, $\sum_{i=1}^{n} (t - x_i)^+ \le \sum_{i=1}^{n} (t - y_i)^+$.

(iii) $\vec{x} \prec \vec{y}$ iff $\forall t \in \mathbb{R}$ $\sum_{j=1}^{n} |x_j - t| \le \sum_{j=1}^{n} |y_j - t|$

$$\left(\sum_{i=1}^{n} \phi_t(x_i) \le \sum_{i=1}^{n} \phi_t(y_i) \right.$$

(no mention of rearrangement). $\qquad \left. \phi_t(r) = |r - t| \right).$

Proof:- $t \in \mathbb{R}$, $\quad x_{k+1}^{\downarrow} \le t \le x_k^{\downarrow}$ $\quad$ (set $x_{n+1}^{\downarrow} = -\infty$)

$$\sum_{j=1}^{n} (x_j - t)^+ = \sum_{j=1}^{k} (x_j^{\downarrow} - t) \le \sum_{j=1}^{k} (y_j^{\downarrow} - t)$$

$$\le \sum_{j=1}^{k} (y_j^{\downarrow} - t)^+ \le \sum_{j=1}^{n} (y_j - t)^+$$

For the converse, pick $t = y_k^{\downarrow}$

$$\sum_{j=1}^{n} (y_j - t)^+ = \sum_{j=1}^{k} (y_j^{\downarrow} - t) = \sum_{j=1}^{k} (y_j^{\downarrow}) - kt$$

$$\sum_{j=1}^{k} (x_j^{\downarrow}) - kt = \sum_{j=1}^{k} (x_j^{\downarrow} - t) \le \sum_{j=1}^{k} (x_j^{\downarrow} - t)^+ \le \sum_{j=1}^{n} (x_j - t)^+$$

$$\leq \sum_{j=1}^{n} (y_j - t)^+ = \sum_{j=1}^{k} (y_j^{\downarrow}) - kt.$$

$$\Rightarrow \sum_{j=1}^{k} (x_j)^{\downarrow} \leq \sum_{j=1}^{k} (y_j^{\downarrow}). \quad \text{(proved)}$$

COR:- If $\vec{x} < \vec{y}$ in $\mathbb{R}^n$, $\vec{u} < \vec{v}$ in $\mathbb{R}^m$, then, $(\vec{x}, \vec{u}) < (\vec{y}, \vec{v})$ in $\mathbb{R}^{n+m}$.

COR:- If $\vec{x} < \vec{y}$ in $\mathbb{R}^n$ & $\alpha \in \mathbb{R}$ then $(\vec{x}, \alpha) < (\vec{y}, \alpha)$

Conversely if $(\vec{x}, \alpha) < (\vec{y}, \alpha)$ for some $\alpha \in \mathbb{R}$, then $\vec{x} < \vec{y}$ in $\mathbb{R}^n$.

~~Theorem~~ $A \in M_n(\mathbb{R})$ is said to be doubly stochastic

if $\sum_{i=1}^{n} a_{ik} = 1 \quad \forall\, 1 \leq k \leq n$ & $\sum_{i=1}^{n} a_{ki} = 1 \quad \forall\, 1 \leq k \leq n$

& $0 \leq a_{ij} \leq 1 \quad \forall\, 1 \leq i, j \leq n$.

Theorem:- $A \in M_n(\mathbb{R})$ is Doubly Stochastic iff $A\vec{x} < \vec{x}$ $\forall\, \vec{x} \in \mathbb{R}^n$. ($\vec{y} < \vec{x} \Rightarrow \exists\, A \in DS_n(\mathbb{R})$ s.t. $\vec{y} = A\vec{x}$)

Proof:- $(\Leftarrow)$ $e_i = (0, \ldots, 0, \overset{i^{th}}{1}, 0, \ldots, 0)$

$$Ae_i < e_i \Rightarrow \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} < \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$0 \leq a_{ni} \leq \cdots \leq a_{1i} \leq 1.$$

vary $1 \leq i \leq n$, we get $\sum_{k=1}^{n} a_{ki} = 1$.

Column sums as 1.

$e = (\frac{1}{n}, \ldots, \frac{1}{n})$

$$\begin{bmatrix} \frac{1}{n} \left(\sum_{i=1}^{n} a_{1i}\right) \\ \vdots \\ \frac{1}{n} \left(\sum_{i=1}^{n} a_{ni}\right) \end{bmatrix} < \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \Rightarrow \sum_{i=1}^{n} a_{ki} = 1 \quad \forall\, 1 \leq k \leq n$$

$\Rightarrow$ Row sums are 1.

$(\Rightarrow)$ A is a Doubly Stochastic Matrix, $\vec{y} = A\vec{x}$.

WLOG, assume $\vec{y}, \vec{x}$ are in decreasing order.

$\rightarrow$ (This is upto left multiplication by a permutation matrix)

$$\sum_{j=1}^{k} y_j = \sum_{i=1}^{k}\left(\sum_{j=1}^{n} a_{ji} x_i\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{k} a_{ij} x_j\right)$$

$$\left[t_i = \sum_{j=1}^{k} a_{ji}\right]$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ji}\right) x_i = \sum_{i=1}^{n} t_i x_i \qquad \left[\sum_{i=1}^{m} t_i = k\right]$$

$$\Rightarrow \sum_{j=1}^{k} y_j - \sum_{j=1}^{k} x_j = \sum_{i=1}^{n} t_i x_i - \sum_{i=1}^{k} x_i \left(+\left(k - \sum_{i=1}^{n} t_i\right) x_k\right)$$

$$= \sum_{i=1}^{n} (t_i - 1)(x_i - x_k) + \sum_{i=k+1}^{m} t_i (x_i - x_k)$$

≤ 0 $[t_i \leq 1] \Rightarrow x_i \leq x_k \ \forall\ k+1 \leq i \leq n]$

[with equality if $k = n$]

$\Rightarrow A\vec{x} < \vec{x}$

T_y-transform ($y \to$ transposition).

$$T_{y,x}\vec{y} = (y_1, \ldots, y_{j-1}, \alpha y_j + (1-\alpha) y_k, \ldots, (1-\alpha) y_j + \alpha y_k, \ldots)$$

$$= \alpha \vec{y} + (1-\alpha)\, T(\vec{y})$$

Doubly Stochastic Matrix.

Theorem:- For $\vec{x}, \vec{y} \in \mathbb{R}^n$, Then The followings are equivalent:

(i) $\vec{x} < \vec{y}$

(ii) $\vec{x}$ is obtained from $\vec{y}$ by $(n-1)$ T_y transform

(iii) $\vec{x} \in$ permutation polytope of $\vec{y}$.

(iv) $\vec{x} = A\vec{y}$ for some Doubly Stochastic matrix A.

Proof:- (i) $\Rightarrow$ (ii) For $n = 2$ $(x_1^{\downarrow}, x_2^{\downarrow}) < (y_1^{\downarrow}, y_2^{\downarrow})$

$x_1 \leq y_1$ $\qquad y_1^{\downarrow} \geq y_2^{\downarrow}$

$x_1 + x_2 = y_1 + y_2$ $\qquad x_1 \geq x_2$.

$\Rightarrow x_2 \geq y_2$.

$$\Rightarrow \frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2}.$$

$x_1 = \alpha y_1 + (1-\alpha) y_2$ $\qquad \alpha \in [0,1]$.

$x_2 = (1-\alpha) y_1 + \alpha y_2$.

$y_k^{\downarrow} \le x_1 \le y_{k-1}^{\downarrow}$ $\quad (y_n^{\downarrow} \le x_1 \le y_1^{\downarrow})$ $\quad x_1 \geq \cdots \geq x_n.$

$x_1 = \alpha y_1^{\downarrow} + (1-\alpha) y_k^{\downarrow}$

$T_{(1,k),\alpha}(\vec{y}) = (x_1, (y_2, \ldots, y_{k-1}, (1-\alpha) y_1^{\downarrow} + \alpha y_k^{\downarrow}, y_{k+1}, \ldots y_n))$

$\qquad\qquad\qquad\qquad\qquad\qquad \vee$

$\qquad\qquad\qquad\qquad\qquad (x_2, \ldots, x_n).$

$y_1 \geq \cdots \geq y_{k-1} \geq x_1 \geq \cdots \geq x_n.$

$$\sum_{j=2}^{m} x_j \le \sum_{j=2}^{m} y_j \qquad 2 \le m \le k-1$$

$k \le m \le n,$
$$\sum_{j=2}^{m} y_j' = \sum_{j=2}^{k-1} y_j + [(1-\alpha) y_1 + \alpha y_k] + \sum_{j=k+1}^{m} y_j$$

$$= \sum_{j=1}^{m} y_j + ((1-\alpha) y_1 + \alpha y_k - y_k - y_1)$$

$$= \sum_{i=1}^{m} y_j - \alpha y_1 + (\alpha - 1) y_k$$

$$= \sum_{i=1}^{n} y_j - x_1 \geq \sum_{j=1}^{m} x_j - x_1 = \sum_{j=2}^{m} x_j$$

$(x_2, \ldots, x_n) < (y_2', \ldots, y_n')$ $\qquad y_j' = y_j, \; j \neq k.$

Hence $\vec{y}$ can be transformed into $\qquad y_k' = (1-\alpha) y_1 + \alpha y_k.$

$\vec{x}$ using at most $(n-2)+1$ T-transform.

(ii) $\Rightarrow$ (iii) is immediate from the fact that T-transform takes $\vec{y}$ to a vector in $K_{\vec{y}}$ $\qquad \cancel{(\alpha\vec{y} + (1-\alpha)} J(\vec{y})$ $\cancel{\in K_{\vec{y}}})$

(& if $\vec{v} \in K_{\vec{y}}$, then $K_{\vec{v}} \subseteq K_{\vec{y}}$). $\qquad (\alpha\vec{y} + (1-\alpha) J(\vec{y}) \in K_{\vec{y}})$.

(iii) $\Rightarrow$ (iv) (by Birkhoff - Von Neumann Theorem)

$T_{J,\alpha}(\vec{v}) = \alpha I \vec{v} + (1-\alpha) P_J \vec{v}$ $\qquad P \to$ permutation matrix.

$\vec{y} \to \cdots \to \vec{x}$

$\alpha_1 P_1(\vec{v}) + \cdots + \alpha_m P_m \vec{v}$ $\qquad \alpha_1 + \cdots + \alpha_m = 1$

$= \underbrace{(\alpha_1 P_1 + \cdots + \alpha_m P_m)}_{\text{Doubly Stochastic.}} \vec{v}$

(iv) $\Rightarrow$ (i) If $A\vec{y} = \vec{x}$, $\quad A\vec{y} < \vec{y} \Rightarrow \vec{x} < \vec{y}$ (by the previous Theorem).

Schur's Theorem:- Let $A \in M_n(\mathbb{C})$ be a Hermitian Matrix.

Let diag(A) denote the vector whose coordinates are the diagonal entries of A. $\lambda(A) \to$ Spectral ~~distance~~ distribution of A. (vector of eigenvalues with multiplicities).

Then $\text{diag}(A) < \lambda(A)$.

Proof:-

$$UAU^* = \text{diag}(\lambda_1, \ldots, \lambda_n) \qquad U \to \text{unitary}.$$

$$\Rightarrow A = U^* \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U.$$

$$\begin{bmatrix} |u_{11}|^2 & |u_{12}|^2 & \cdots & \\ \vdots & & \ddots & \\ \vdots & & & |u_{nn}|^2 \end{bmatrix}$$

$\to$ Doubly Stochastic Matrix

$$U^*\boxed{\lambda}\,U = \begin{bmatrix} \lambda_1 \overline{u_{11}} & \cdots & \lambda_n \overline{u_{n1}} \\ \vdots & & \vdots \\ \lambda_1 \overline{u_{1n}} & & \lambda_n \overline{u_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 |u_{11}|^2 + \cdots + \lambda_n |u_{n1}|^2 & & * \\ & \ddots & \\ * & & \lambda_1 |u_{1n}|^2 + \cdots + \lambda_n |u_{nn}|^2 \end{bmatrix}$$

$$\text{Diag}(A) = \begin{bmatrix} |u_{11}|^2 & \cdots & |u_{n1}|^2 \\ |u_{12}|^2 & \cdots & |u_{n2}|^2 \\ \vdots & & \vdots \\ |u_{1n}|^2 & \cdots & |u_{nn}|^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$\hookrightarrow$ Doubly Stochastic Matrix

$\Rightarrow \text{Diag}(A) < (\lambda_1, \ldots, \lambda_n)$.

Done by the Previous Theorem.

Horn's Theorem If $\vec{v} < \vec{\omega}$, there is a Hermitian matrix in $M_n(\mathbb{C})$ with spectral distribution is $\vec{\omega}$ and diagonal $\vec{v}$.

Proof:- $A = \begin{bmatrix} a_{11} & \bar{a}_{12} \\ a_{12} & a_{22} \end{bmatrix}$ $a_{12} = re^{i\theta}$ $\xi = e^{i(\pi/2 - \theta)}$

$$U \to \begin{bmatrix} \xi \sin\theta & -\cos\theta \\ \xi \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} a_{11} & \bar{a}_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \bar{\xi} \sin\theta & \bar{\xi} \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

$$\operatorname{diag}(UAU^*) = \left((a_{11} \sin^2\theta + a_{22} \cos^2\theta), (a_{11} \cos^2\theta + a_{22} \sin^2\theta) \right).$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

If $\vec{v}$ is the diagonal of a Hermitian matrix with eigenvalue distribution $\vec{\omega}$, then so is every T-transform of $\vec{v}$.