02-Sept-2022 Friday

Lecture - 07

Adv. Linear Algebra

Review

→ We had $X_{ij} \sim N(0,1)$, $1 \le i,j \le n$ had joint density

$$\frac{1}{(2\pi)^{n^2/2}} \exp\left(-\frac{1}{2}\|X\|_F^2\right)$$

→ For $M \in O_n(\mathbb{R})$, $X \longmapsto MX$ then $X \overset{d}{=} MX$ $(X = (x_{ij}))$ [Fröbenius norm unitary invar.]

Also $m(\{A \in M_n(\mathbb{R}) : \det(A) = 0\}) = 0$; so $A \in GL_n(\mathbb{R})$ almost surely

so define a map, $T : GL_n(\mathbb{R}) \longrightarrow O_n(\mathbb{R})$

by $[x_1 | \cdots | x_n] \longmapsto \left[\frac{x_1}{\|x_1\|} \,\Big|\, \frac{x_2 - \langle x_2, x_1\rangle x_1}{\|x_2 - \langle x_2, x_1\rangle x_1\|} \,\Big|\, \cdots\right]$

* we claim that $MT(X) = TM(X)$, $M \in O_n(\mathbb{R})$

$x_2 \longmapsto x_2 - \langle x_2, x_1\rangle x_1$

$\Rightarrow Mx_2 \longmapsto Mx_2 - \langle x_2, x_1\rangle Mx_1$

$= Mx_2 - \langle Mx_2, Mx_1\rangle Mx_1$ ($\because M$ orthogonal)

$\therefore MT(X) = T(MX) \overset{d}{=} T(X)$

* $T(X)$ gives a left $O_n(\mathbb{R})$ invariant distribution on $O_n(\mathbb{R})$

If we used QR-decomposition then the Q need not be left $O_n(\mathbb{R})$-invariant.

Remarks: (i) X be random matrix of iid standard Gaussian r.v's & $X = QR$, its QR-decomposition obtained via Gram-Schmidt then Q is haar distributed on the orthogonal group

(ii) Note that X^tX is positive-definite, so we have square root

$U := X(X^tX)^{-1/2}$ $\quad (X = U(X^tX)^{1/2})$

so $U^tU = ((X^tX)^{-1/2}X^t)(X(X^tX)^{-1/2})$

$= I_n$

Now for a fixed $V \in O_n(\mathbb{R})$ we know $VX \overset{d}{=} X$

~~$VX = VX($~~ $\quad VX \overset{d}{=} X$

$VX((VX)^t(VX))^{-1/2} \qquad X(X^tX)^{-1/2}$

$= VX(X^tX)^{-1/2}$

$X \longrightarrow VX$, $X \overset{d}{=} VX$; $U \longrightarrow VU$, $U \overset{d}{=} VU$ fixed matrix, random matrix

Distribution of U is left-invariant under $O_n(\mathbb{R})$

Polar Decomposition

For every matrix $X \in M_n(\mathbb{C})$ there is a unique positive semi-definite matrix P and a partial isometry V (with minimal rank) such that $X = VP$

If X is invertible, $P = (X^*X)^{1/2}$, $V = X(X^*X)^{-1/2}$

Note that

$$\mathcal{N}((X^*X)^{1/2}) = N(X)$$

Indeed, $(X^*X)^{1/2} v = 0 \iff Xv = 0$

$$\Updownarrow$$

$$\langle (X^*X)^{1/2} v, (X^*X)^{1/2} v \rangle = 0$$

$$\Updownarrow$$

$$\langle X^*X v, v \rangle = 0 \iff \langle Xv, Xv \rangle = 0 . \quad \square$$

So we get $\mathcal{R}((X^*X)^{1/2}) = \mathcal{R}(X^*)$.

Def $V \in M_n(\mathbb{C})$ is called a partial isometry if V^*V is an orthogonal projection i.e. $(V^*V)^2 = V^*V =: E$

E called initial projection. V is an isometry from $Ran(E)$ to $Ran(VV^*)$ ~~[Verify these are projection]~~

* E is a projection then note that $rank(E) = tr(E)$ [both are unitary invariant, put E in diagonal form]

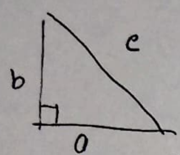
* $V' : \mathcal{N}(X^*) \longrightarrow \mathcal{N}((X^*X)^{1/2})$ then $V + V'$ is unitary
* When X^*X is invertible then polar decomposition is unique as, $X = UP \Rightarrow X^*X = P^2 \Rightarrow P = \sqrt{X^*X}$ Thus $U = X(\sqrt{X^*X})^{-1}$

* What about ~~non~~ singular case? We will see next class.

Example: (isometry

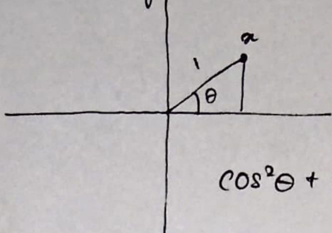
The pythagorean theorem

* converse is called Carpenter's theorem : they want to figure out right angle or not.: "if $a^2+b^2=c^2$ then Δ is right."



$c^2 = a^2+b^2$

version 1



$\cos^2\theta + \sin^2\theta = 1$

Version 2

$|\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 = 1$

$\Updownarrow$

$|\langle e_1, x\rangle|^2 + |\langle e_2, x\rangle|^2 = 1$

Version 3

i.e. if a_1 = length of proj of e_1 on span(x) & a_2 = length of proj of e_2 on span(x)

then, $a_1^2 + a_2^2 = 1$

Version 4 ~~X be a~~ $\mathcal{H}$ be a complex hilbert space & $\{e_i\}_{i\in I}$ are orthonormal basis then Parseval's theorem says

$$\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 \qquad \text{[KR - chap 2]}$$

Version 5 (analog to version 3) $\{e_i\}_{i\in I}$ orthonormal basis of $\mathcal{H}$ then sum of squares of length of orthogonal projections of each e_i on every 1-Dim subspace is 1.

Carpenter's theorem for ver. 05: $(t_i)_{i\in I}$ with $t_i \in \mathbb{R}_{\geq 0}$ such that

$\sum_{i\in I} t_i^2 = 0$ then $\sum_{i\in I} t_i e_i = x$ generate a 1-Dim subspace of $\mathcal{H}$

~~on~~ on which e_i has a projection of length t_i. i.e. $\langle x, e_i\rangle = t_i$

Version 6 : if $\{e_i\}_{i\in I}$ an orthonormal basis of $\mathcal{H}$ then the sum of squares of lengths of the orthogonal proj of each e_i on any given m-dim subspace of $\mathcal{H}$ is m. ($m \in \mathbb{N}$; ~~for #(m) ≥ ℵ₀ ?~~)

Proof: W be a m-dim subspace & $\{f_1, \dots, f_m\}$ be orthonormal basis of W. P_W be projection onto W. Then

$i \in I$; $P_W e_i = \langle P_W e_i, f_1\rangle f_1 + \dots + \langle P_W e_i, f_m\rangle f_m = \sum_{j=1}^{m} \langle P_W e_i, f_j\rangle f_j$

$$\therefore \sum_{i\in I} \|P_W e_i\|^2 = \sum_{i\in I}\sum_{j=1}^{m} |\langle e_i, f_j\rangle|^2 = \sum_{j=1}^{m}\sum_{i\in I} |\langle e_i, f_j\rangle|^2 \quad (\because \text{terms are} \geq 0 \text{ so sum swap})$$

$$= \sum_{j=1}^{m} 1 = m \qquad \text{(by Parseval)} \qquad \square$$

Version.07. [Matrix version] $V \subseteq \mathbb{C}^n$ $\{e_1, \ldots, e_n\}$ standard orthonormal basis of $\mathbb{C}^n$. $V \subseteq \mathbb{C}^n$ be an m-dim subspace. $E_V : \mathbb{C}^n \to \mathbb{C}^n$ be orthogonal projection onto V. Suppose as a matrix $E_V = (a_{ij})_{1 \le i,j \le n}$ then

$$a_{ii} = \langle E_V e_i, e_i \rangle = \langle E_V e_i, E_V e_i \rangle = [\because E_V \text{ projection}]$$

Thus $a_{ii} = \| E_V e_i \|^2$.

"The trace of a projection with m-dimensional range (i.e. rank m) is m" i.e.

$$tr(E_V) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \| E_V e_i \|^2 = m$$

Carpenter's theorem Given $\{a_{ii}\}_{i=1}^{n}$ such that

$$\begin{bmatrix} a_{11} & & * & \\ & a_{22} & & \\ * & & \ddots & \\ & & & a_{nn} \end{bmatrix}$$

$0 \le a_{ii} \le 1$ and, $\sum_{i=1}^{n} a_{ii} \in \mathbb{Z}_+$

can we form a matrix (filling all rest entries so as to make matrix orthogonal) such that matrix becomes orthogonal projection?

[Kadison – The pythagorean theorem I, (2001)]

For proving this we need majorisation.

* $D_m = diag(\underbrace{1, \ldots, 1}_{m\text{-times}}, \underbrace{0 \ldots, 0}_{n-m \text{ times}})$ then every rank m projection is unitarily equivalent to ~~diag~~ D_m. $\Phi : M_n(\mathbb{C}) \to Diag_n(\mathbb{C})$
$Im(\Phi|_{\{UEU^* : U \in U_n(\mathbb{C})\}})$ is a convex polytope

* Schur- Horn Theorem:

H ~~be~~ be a Hermitian matrix with spectral distribution $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$Img\left(\Phi\Big|_{\{UHU^* : U \in U_n(\mathbb{C})\}}\right)$$ is a convex set / polytope generated by $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

Majorization (Ry: Bhatia)

Def Let $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ & $\vec{x}^{\downarrow}$ ($\vec{x}^{\uparrow}$ resp.) be ~~elements~~ vectors in $\mathbb{R}^n$ obtained by ordering coordinates in decreasing (increasing resp.) order.
i.e. $x^{\downarrow} = (x_1^{\downarrow}, \dots, x_n^{\downarrow})$ with $x_i^{\downarrow}$ be permuted x_i's with $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \dots \geq x_n^{\downarrow}$
(Similarly for $x^{\uparrow}$). (Note: $x_i^{\downarrow} = x_{n-i}^{\uparrow}$)

We say that $x \in \mathbb{R}^n$ is majorized by $y \in \mathbb{R}^n$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad ; \ \forall\ 1 \leq k < n$$

&

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow} \quad . \text{ Notation: } x \precsim y$$

[Wealth distn interpretation ...]

Defn $K_{\vec{x}}$ be the closed convex hull of $\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in S_n\}$
Then $K_{\vec{x}}$ is called the permutation polytope generated by $\vec{x}$.

* Note that its a compact convex set.

Theorem: $\vec{x}, \vec{y} \in \mathbb{R}^n$ & $x_1 \geq x_2 \geq \dots \geq x_n$ & $y_1 \geq y_2 \geq \dots \geq y_n$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Then the following are equivalent:

(i) $(x_1, \dots, x_n) = \vec{x} \in K_{\vec{y}}$

(ii) $\vec{x} \precsim \vec{y}$ (i.e. y majorizes x)

—x———x———